Functions

• Section 2.3

**Definition:** Let $A$ and $B$ be nonempty sets. A function $f$ from $A$ to $B$, denoted $f: A \rightarrow B$, is an assignment of each element of $A$ to exactly one element of $B$. We write $f(a) = b$ if $b$ is the unique element of $B$ assigned by the function $f$ to the element $a$ of $A$.

• Functions are sometimes called mappings or transformations.

Is each element of $B$ assigned only once?

<table>
<thead>
<tr>
<th>Students</th>
<th>Grades</th>
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<tbody>
<tr>
<td>Carlota Rodriguez</td>
<td>A</td>
</tr>
<tr>
<td>Sandeep Patel</td>
<td>B</td>
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<tr>
<td>Jalen Williams</td>
<td>C</td>
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<tr>
<td>Kathy Scott</td>
<td>D</td>
</tr>
</tbody>
</table>

Functions

• A function $f: A \rightarrow B$ can also be defined as a subset of $A \times B$ (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.

Specifically, a function $f$ from $A$ to $B$ contains one, and only one ordered pair $(a, b)$ for every element $a \in A$.

$\forall x [x \in A \rightarrow \exists y [y \in B \land (x, y) \in f]]$

and

$\forall x \forall y_1 \forall y_2 [((x, y_1) \in f \land (x, y_2) \in f) \rightarrow y_1 = y_2]$

What do the logical expressions say?

Representing Functions

• Functions may be specified in different ways:
  – An explicit statement of the assignment.
    Previous students and grades example.
  – A formula.
    $f(x) = x + 1$
  – A computer program.
    • A Java program that when given an integer $n$, produces the $n$th Fibonacci Number (covered later)

Questions

$f(a) = ?$

The image of d is ?

The domain of $f$ is ?

The codomain of $f$ is ?

The preimage of y is ?

$f(A) = ?$

The preimage(s) of z is (are) ?
Question on Functions and Sets

- If \( f : A \rightarrow B \) and \( S \) is a subset of \( A \), then \( f(S) = \{ f(s) | s \in S \} \)

\[ f\{a,b,c,\} \text{ is ?} \]
\[ f\{c,d\} \text{ is ?} \]

Injections

**Definition**: A function \( f \) is said to be **one-to-one**, or **injective**, if and only if \( f(a) = f(b) \) implies that \( a = b \) for all \( a \) and \( b \) in the domain of \( f \). A function is said to be an injection if it is one-to-one.

Surjections

**Definition**: A function \( f \) from \( A \) to \( B \) is called **onto** or **surjective**, if and only if for every element \( b \in B \) there is an element \( a \in A \) with \( f(a) = b \). A function \( f \) is called a surjection if it is onto.

Bijections

**Definition**: A function \( f \) is a **one-to-one correspondence**, or a bijection, if it is both one-to-one and onto (surjective and injective).
Showing that \( f \) is one-to-one or onto

Suppose that \( f : A \to B \).

To show that \( f \) is injective  
Show that if \( f(x) = f(y) \) for arbitrary \( x, y \in A \) with \( x \neq y \), then \( x = y \).

To show that \( f \) is not injective  
Find particular elements \( x, y \in A \) such that \( x \neq y \) and \( f(x) = f(y) \).

To show that \( f \) is surjective  
Consider an arbitrary element \( y \in B \) and find an element \( x \in A \) such that \( f(x) = y \).

To show that \( f \) is not surjective  
Find a particular \( y \in B \) such that \( f(x) \neq y \) for all \( x \in A \).

Showing that \( f \) is one-to-one or onto

**Example 1:** Let \( f \) be the function from \( \{a,b,c,d\} \) to \( \{1,2,3\} \) defined by \( f(a) = 3, f(b) = 2, f(c) = 1, \) and \( f(d) = 3 \). Is \( f \) one-to-one? Is \( f \) an onto function?

**Solution:**

**Example 2:** Is the function \( f(x) = x^2 \) from the set of integers one-to-one? Is \( f \) onto?

**Solution**

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**Horizontal line test**

- If a horizontal line intersects \( f \) in more than one point, then not 1-1

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**Inverse Functions**

**Definition:** Let \( f \) be a bijection from \( A \) to \( B \). Then the *inverse* of \( f \), denoted \( f^{-1} \), is the function from \( B \) to \( A \) defined as

\[
f^{-1}(y) = x \iff f(x) = y
\]

No inverse exists unless \( f \) is a bijection. Why?
Inverse Functions

Example 1: Let $f$ be the function from $\{a, b, c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is $f$ invertible and if so what is its inverse?

Solution: The function $f$ is invertible because it is a one-to-one correspondence. The inverse function $f^{-1}$ reverses the correspondence given by $f$, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Example 2: Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be such that $f(x) = 2x + 1$. Is $f$ invertible, and if so, what is its inverse?

Solution: $f$ is NOT invertible since it is not onto.

$f^{-1}(y) = (y - 1)/2$.

Example 2 again: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be such that $f(x) = 2x + 1$. Is $f$ invertible?

Solution: The function $f$ is invertible because it is a one-to-one correspondence.

Example 3: Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be such that $f(x) = x^2$. Is $f$ invertible, and if so, what is its inverse?

Solution: $f$ is not invertible because it is not one-to-one.
Composition

**Definition:** Let \( f: B \to C \), \( g: A \to B \). The composition of \( f \) with \( g \), denoted \( f \circ g \), is the function from \( A \) to \( C \) defined by

\[
f \circ g(x) = f(g(x))
\]

**Example 1:** If \( f(x) = x^2 \) and \( g(x) = 2x + 1 \) then

\[
f(g(x)) =
\]

and \( g(f(x)) = \)

**Composition Questions**

**Example 2:** Let \( g \) be the function from the set \( \{a,b,c\} \) to itself such that \( g(a) = b, g(b) = c, \) and \( g(c) = a \). Let \( f \) be the function from the set \( \{a,b,c\} \) to the set \( \{1,2,3\} \) such that \( f(a) = 3, f(b) = 2, \) and \( f(c) = 1 \). What is the composition of \( f \) and \( g \), and what is the composition of \( g \) and \( f \).
Composition Questions

**Example 3**: Let \( f \) and \( g \) be functions from the set of integers to the set of integers defined by \( f(x) = 2x + 3 \) and \( g(x) = 3x + 2 \). What is the composition of \( f \) and \( g \), and also the composition of \( g \) and \( f \)?

Solution:

\[
\begin{align*}
f \circ (g(x)) &= f\left(\left[3x + 2\right]\right) \\
&= \left[3(3x + 2) + 3\right] \\
&= \left[9x + 9\right] \\
&= 3x + 3
\end{align*}
\]

\[
\begin{align*}
g \circ (f(x)) &= g\left(\left[2x + 3\right]\right) \\
&= \left[3(2x + 3) + 2\right] \\
&= \left[6x + 11\right] \\
&= 2x + 3 + 1
\end{align*}
\]

Graphs of Functions

- Let \( f \) be a function from the set \( A \) to the set \( B \). The **graph** of the function \( f \) is the set of ordered pairs \( \{(a, b) | a \in A \text{ and } f(a) = b\} \).

Some Important Functions

- The **floor** function, denoted \( \lfloor x \rfloor \) is the largest integer less than or equal to \( x \).

- The **ceiling** function, denoted \( \lceil x \rceil \) is the smallest integer greater than or equal to \( x \).

**Example**: \( \lfloor 3.5 \rfloor = 4 \quad \lceil 3.5 \rceil = 3 \)

\( \lfloor -1.5 \rfloor = -1 \quad \lceil -1.5 \rceil = -2 \)

Floor and Ceiling Functions

- Graph of \( f(n) = 2n + 1 \) from \( \mathbb{Z} \) to \( \mathbb{Z} \)
- Graph of \( f(x) = x^2 \) from \( \mathbb{Z} \) to \( \mathbb{Z} \)

**Graph of (a) Floor and (b) Ceiling Functions**
**Floor and Ceiling Functions**

| TABLE 1 Useful Properties of the Floor and Ceiling Functions. |
|---|---|
| (1a) [x] = n if and only if n ≤ x < n + 1 |
| (1b) [x] = n if and only if n - 1 < x ≤ n |
| (1c) [x] = n if and only if x - 1 < n ≤ x |
| (1d) [x] = n if and only if x ≤ n < x + 1 |
| (2) x - 1 < [x] ≤ x ≤ [x] < x + 1 |

**Example:** Prove that if x is a real number, then [2x] = [x] + [x + 1/2]

**Factorial Function**

**Definition:** f: N → Z⁺, denoted by f(n) = n!, is the product of the first n positive integers when n is a nonnegative integer.

\[ f(n) = 1 \cdot 2 \cdots (n - 1) \cdot n, \quad f(0) = 0! = 1 \]

**Examples:**

\[ f(1) = 1! = 1 \]
\[ f(2) = 2! = 1 \cdot 2 = 2 \]
\[ f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720 \]
\[ f(20) = 2,432,902,008,176,640,000. \]

**Partial Functions**

**Definition:** A partial function f from a set A to a set B is an assignment to each element a in a subset of A, called the domain of definition of f, of a unique element b in B.

- The sets A and B are called the domain and codomain of f, respectively.
- We say that f is undefined for elements in A that are not in the domain of definition of f.
- When the domain of definition of f equals A, we say that f is a total function.

**Example:** f: Z → R where f(n) = √n is a partial function from Z to R where the domain of definition is the set of nonnegative integers. Note that f is undefined for negative integers.