Sequences and Summations

• Section 2.4

Introduction

• Sequences are ordered lists of elements.
  – 1, 2, 3, 5, 8
  – 1, 3, 9, 27, 81, .......

• Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.

• We will introduce the terminology to represent sequences and sums of the terms in the sequences.
Sequences

**Definition:** A sequence is a function from a subset of the integers (usually either the set \{0, 1, 2, 3, 4, ……\} or \{1, 2, 3, 4, ……\}) to a set \(S\).

- The notation \(a_n\) is used to denote the image of the integer \(n\). We can think of \(a_n\) as the equivalent of \(f(n)\) where \(f\) is a function from \{0,1,2,…..\} to \(S\). We call \(a_n\) a term of the sequence.

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Geometric Progression

**Definition:** A geometric progression is a sequence of the form:

\[ a, ar, ar^2, \ldots, ar^n, \ldots \]

where the initial term \(a\) and the common ratio \(r\) are real numbers.

**Examples:**

1. Let \(a = 1\) and \(r = 3\). Then the sequence is:

\[ \{1, 3, 9, 27, 81, \ldots\} \]

2. What are \(a\) and \(r\) for this sequence?

\[ \{2, -4, 8, -16, 32, \ldots\} \]

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Arithmetic Progression

**Definition:** An arithmetic progression is a sequence of the form:

\[ a, a + d, a + 2d, \ldots, a + nd, \ldots \]

where the initial term \(a\) and the common difference \(d\) are real numbers.

**Examples:**

1. Let \(a = 2\) and \(d = 3\):

\[ \{2, 5, 8, 11, 14, \ldots\} \]

2. What are \(a\) and \(d\) for this sequence?

\[ \{1, 1/2, 0, -1/2, -1, \ldots\} \]
Strings

**Definition**: A *string* is a finite sequence of characters from a finite set (an alphabet).
- Sequences of characters or bits are important in computer science.
- The *empty string* is represented by $\lambda$.
- The string $abcde$ has length 5.

### Recurrence Relations

**Definition**: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses $a_n$ in terms of one or more of the previous terms of the sequence, namely, $a_0$, $a_1$, ..., $a_{n-1}$, for all integers $n$ with $n \geq n_0$, where $n_0$ is a nonnegative integer.
- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

#### Questions about Recurrence Relations

**Example 1**: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, 4, ...$ and suppose that $a_0 = 2$. What are $a_1$, $a_2$ and $a_3$? [Here $a_0 = 2$ is the initial condition.]

**Example 2**: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, ...$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are $a_2$ and $a_3$? [Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.]
Fibonacci Sequence

**Definition:** Define the *Fibonacci sequence*, \( f_0, f_1, f_2, \ldots \), by:
- Initial Conditions: \( f_0 = 0, f_1 = 1 \)
- Recurrence Relation: \( f_n = f_{n-1} + f_{n-2} \)

**Example:** Find \( f_2, f_3, f_4, f_5 \) and \( f_6 \).

**Answer:**
\[
\begin{align*}
  f_2 &= f_1 + f_0 = 1 + 0 = 1, \\
  f_3 &= f_2 + f_1 = 1 + 1 = 2, \\
  f_4 &= f_3 + f_2 = 2 + 1 = 3, \\
  f_5 &= f_4 + f_3 = 3 + 2 = 5, \\
  f_6 &= f_5 + f_4 = 5 + 3 = 8.
\end{align*}
\]

The Rabbit problem by Fibonacci

- You have one pair of rabbits just born
- After one month, the female is mature and now the pair will give birth to a female and male rabbit in one month and then every month.
- How many rabbits do you have after six months?

Formally, Fibonacci Sequence

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\end{align*}
\]
Solving Recurrence Relations

Finding a formula for the $n$th term of the sequence generated by a recurrence relation is called solving the recurrence relation. Such a formula is called a closed formula. Various methods for solving recurrence relations will be covered later.

Iterative Solution Example

- **Method 1:** Working upward, forward substitution
  - Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2, 3, 4, \ldots$ and suppose that $a_1 = 2$
  - $a_2 = a_1 + 3 = 2 + 3$

Financial Application

**Example:** Suppose that a person deposits $12,200 in a savings account at a bank yielding 6% per year with interest compounded annually. How much will be in the account after 10 years, 20 years, 30 years?

Let $P_n$ denote the amount in the account after $n$ years. $P_n$ satisfies the following recurrence relation:

\[
P_n = P_{n-1} + 0.06P_{n-1} = (1.06)P_{n-1}
\]

with the initial condition $P_0 = 10,000$
Solution – Forward Substitution

- \( P_n = 1.06 \times P_{n-1} \) with \( P_0 = 10,000 \)
- \( P_1 = 1.06 \times P_0 \)

Examples with Sequences

**Example 1:** Find formulae for the sequences with the following first five terms: 1, \( \frac{1}{2} \), \( \frac{1}{4} \), \( \frac{1}{8} \), \( \frac{1}{16} \),

**Example 2:** Consider 1, 3, 5, 7, 9

**Example 3:** 1, -1, 1, -1, 1

Special Integer Sequences

Given a few terms of a sequence, try to identify the sequence. Conjecture a formula, recurrence relation, or some other rule.

Some questions to ask?

- Are there repeated terms of the same value?
- Can you obtain a term from the previous term by adding an amount or multiplying by an amount?
- Can you obtain a term by combining the previous terms in some way?
- Are they cycles among the terms?
- Do the terms match those of a well known sequence?

Useful Sequences

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Some Useful Sequences.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n^2 )</td>
<td>1, 4, 9, 16, 25, 36, 49, 64, 81, 100, …</td>
</tr>
<tr>
<td>( n^3 )</td>
<td>1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, …</td>
</tr>
<tr>
<td>( n^4 )</td>
<td>1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, …</td>
</tr>
<tr>
<td>( 2^n )</td>
<td>2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, …</td>
</tr>
<tr>
<td>( 3^n )</td>
<td>3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, …</td>
</tr>
<tr>
<td>( n! )</td>
<td>1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, …</td>
</tr>
<tr>
<td>( f_n )</td>
<td>1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …</td>
</tr>
</tbody>
</table>
Guessing Sequences

**Example:** Conjecture a simple formula for $a_n$ if the first 10 terms of the sequence $\{a_n\}$ are 1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047.

**Solution:** Note each term is about 3 times previous term. Compare with sequence $\{b_n\}$. Notice the $n$th term is $\frac{1}{3}$ less than the corresponding power of 3. So a good conjecture is $a_n = 3^n - \frac{1}{3}$. 

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**Summations**

- More generally for a set $S$:

$$\sum_{j \in S} a_j$$

- Examples:

$$r^0 + r^1 + r^2 + r^3 + \cdots + r^n = \sum_{j=0}^{n} r^j$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{i=1}^{\infty} \frac{1}{i}$$

If $S = \{2, 5, 7, 10\}$ then $\sum_{j \in S} a_j = a_2 + a_5 + a_7 + a_{10}$.

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**Product Notation**

- More generally for a set $S$:

$$\prod_{j \in S} a_j$$

- Examples:

$$r^0 \times r^1 \times r^2 \times r^3 \times \cdots \times r^n = \prod_{j=0}^{n} r^j$$

$$1 \times \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \cdots = \prod_{i=1}^{\infty} \frac{1}{i}$$

If $S = \{2, 5, 7, 10\}$ then $\prod_{j \in S} a_j = a_2 \times a_5 \times a_7 \times a_{10}$.
Example

- Find a formula for $\sum_{k=1}^{n} 2k - 1$
- $1 + 3 + 5 + 7 + \ldots (2n-1)$

Example solve (cont)

- $a_n = a_{n-1} + 2n - 1$

Some Useful Summation Formulae

<table>
<thead>
<tr>
<th>Sum</th>
<th>Closed Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum_{k=0}^{n} a r^k$ (r ≠ 0)</td>
<td>$\frac{a r^{n+1} - a}{r - 1}$, $r \neq 1$</td>
</tr>
<tr>
<td>$\sum_{k=1}^{n} k$</td>
<td>$\frac{n(n + 1)}{2}$</td>
</tr>
<tr>
<td>$\sum_{k=1}^{n} k^2$</td>
<td>$\frac{n(n + 1)(2n + 1)}{6}$</td>
</tr>
<tr>
<td>$\sum_{k=1}^{n} k^3$</td>
<td>$\frac{n^2(n + 1)^2}{4}$</td>
</tr>
<tr>
<td>$\sum_{k=0}^{\infty} x^k$, $</td>
<td>x</td>
</tr>
<tr>
<td>$\sum_{k=1}^{\infty} k x^{k-1}$, $</td>
<td>x</td>
</tr>
</tbody>
</table>

Geometric Series: We just proved this.

Later we will prove some of these by induction.

Proof in text (requires calculus)