Chap 2.5 Cardinality
How many elements? Can you list them in an ordered way so you don’t miss any of them?

- \( \{x \in Z | x \mod 5 = 0, \text{ and } 0 < x \leq 100\} \)
- \( \{x \in Z | x \mod 5 = 0\} \)
- All the subsets of \( \{x \in Z | x \mod 5 = 0\} \)

Test 1
- Closed book, closed notes, closed neighbor
- There will be a handout of formulas supplied
- Topics:
  - Reading covers - Chap. 1, 2, and 13.3
  - Logic, Sets, Functions, Sequences, Cardinality, DFA

Announcements
- Exam 1 is Tuesday, Oct. 1
- No class, Oct 3, No recitation Oct 4-7
- Prof. Rodger is out Sep 30-Oct 4
- There is Recitation: Sept 27-30.

CompSci 230
Discrete Math for Computer Science

\[
\begin{bmatrix}
1 & 1 & 1 \\
-1 & 0 & 1 \\
1 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
J \\
M \\
S
\end{bmatrix}
= 
\begin{bmatrix}
100 \\
40 \\
0
\end{bmatrix}
\]

Sep 26, 2013

Prof. Rodger

Slides modified from Rosen
Cardinality

**Definition:** The cardinality of a set $A$ is equal to the cardinality of a set $B$, denoted $|A| = |B|$, if and only if there is a one-to-one correspondence (i.e., a bijection) from $A$ to $B$.

- If there is a one-to-one function (i.e., an injection) from $A$ to $B$, the cardinality of $A$ is less than or the same as the cardinality of $B$ and we write $|A| \leq |B|$.

- When $|A| \leq |B|$ and $A$ and $B$ have different cardinality, we say that the cardinality of $A$ is less than the cardinality of $B$ and write $|A| < |B|$.

**Showing that a Set is Countable**

- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).

- The reason for this is that a one-to-one correspondence $f$ from the set of positive integers to a set $S$ can be expressed in terms of a sequence $a_1,a_2,...,a_n,...$ where $a_1 = f(1)$, $a_2 = f(2)$, ..., $a_n = f(n)$, ...

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**Hilbert’s Grand Hotel**

The Grand Hotel (example due to David Hilbert) has countably infinite number of rooms, each occupied by a guest. We can always accommodate a new guest at this hotel. How is this possible?

**Explanation:** Because the rooms of Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on. When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general the guest in Room $n$ to Room $n+1$, for all positive integers $n$. This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms.

The hotel can also accommodate a countable number of new guests, all the guests on a countable number of buses where each bus contains a countable number of guests.
Showing that a Set is Countable

**Example 1:** Show that the set of positive even integers $E$ is a countable set.

**Solution:** Let $f(x) = 2x$.

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & \ldots \\
\hline
2 & 4 & 6 & 8 & 10 & 12 & \ldots \\
\end{array}
\]

Then $f$ is a bijection from $\mathbb{N}$ to $E$ since $f$ is both one-to-one and onto. To show that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n = 2m$, and so $n = m$. To see that it is onto, suppose that $t$ is an even positive integer. Then $t = 2k$ for some positive integer $k$ and $f(k) = t$.

The Positive Rational Numbers are Countable

**Definition:** A rational number can be expressed as the ratio of two integers $p$ and $q$ such that $q \neq 0$.

- $\frac{3}{4}$ is a rational number
- $\sqrt{2}$ is not a rational number.

**Example 3:** Show that the positive rational numbers are countable.

**Solution:** The positive rational numbers are countable since they can be arranged in a sequence:

$r_1, r_2, r_3, \ldots$

Does this work?

**For p/q**

- List all those numbers with $q=1$
  - $1/1, 2/1, 3/1, 4/1, 5/1, \ldots$
- Then list all those numbers with $q=2$
  - $1/2, 2/2, 2/3, 2/4, 2/5, \ldots$
- Then list all those numbers with $q=3$, etc
Solution for listing out all p/q

- List all s.t. \( p+q=2 \)
- Then all s.t. \( p+q=3 \)
- Then all s.t. \( p+q=4 \)
- Etc
- Why does this work?

Example - Strings

- Show that the set \( S \) of strings over the alphabet \{0,1\} is countable.
- What is \( S \)? Is \( S \) infinite?

Solution:

Example: Is the set of all Java programs countable?

- \( \Sigma \) = the set of all symbols in Java programs
- \( S \) is the set of all strings over the alphabet

The Real numbers are uncountable

- Proof(sketch) by diagonalization
- Suppose we can list out all the real numbers without missing any of them.
• List out all real numbers
• Claim we missed one!
• There is a number whose \( i \)th digit is different from the \( i \)th digit in the \( i \)th number.

\[
\begin{align*}
1 & \leftrightarrow 0 \cdot 3 9 7 2 0 4 8 1 7 \ldots \\
2 & \leftrightarrow 0 \cdot 5 2 6 6 1 3 8 0 9 \ldots \\
3 & \leftrightarrow 0 \cdot 4 9 8 3 1 0 1 2 3 \ldots \\
4 & \leftrightarrow 0 \cdot 2 7 5 4 1 8 8 3 1 \ldots \\
5 & \leftrightarrow 0 \cdot 0 0 2 2 0 0 0 2 5 \ldots \\
6 & \leftrightarrow 0 \cdot 9 9 9 9 0 4 6 8 1 \ldots \\
\vdots &
\end{align*}
\]

Matrices

• Examples:
  – Graph Theory - Express which vertices of a graph are connected by edges
  – Graphics
    • Represent a 3D object with a matrix
    • Project a 3D object onto a 2D screen
    • Optimal curve fitting
  – Transportation systems.
  – Communication networks
  – Economics and Game Theory
• For now, definition and basic operations

Matrix

**Definition:** A *matrix* is a rectangular array of numbers. A matrix with \( m \) rows and \( n \) columns is called an \( m \times n \) matrix.

  – The plural of matrix is *matrices*.
  – A matrix with the same number of rows as columns is called *square*.
  – Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

\[
\begin{bmatrix}
1 & 1 \\
0 & 2 \\
1 & 3 \\
\end{bmatrix}
\]

**Notation**

• Let \( m \) and \( n \) be positive integers and let

\[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

• The \( i \)th row of \( A \) is the \( 1 \times n \) matrix \([a_{i1}, a_{i2}, \ldots, a_{in}]\).
• The \( j \)th column of \( A \) is the \( m \times 1 \) matrix:

\[
\begin{bmatrix}
a_{1j} \\
a_{2j} \\
\vdots \\
a_{mj}
\end{bmatrix}
\]

• The \((i,j)\)th element or entry of \( A \) is the element \( a_{ij} \). We can use \( A = [a_{ij}] \) to denote the matrix with its \((i,j)\)th element equal to \( a_{ij} \).
Matrix Arithmetic: Addition

**Definition:** Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices. The sum of $A$ and $B$, denoted by $A + B$, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its $(i,j)$th element. In other words,

$$A + B = [a_{ij} + b_{ij}].$$

**Example:**

$$
\begin{bmatrix}
1 & 0 & -1 \\
2 & 2 & -3 \\
3 & 4 & 0
\end{bmatrix} + 
\begin{bmatrix}
3 & 4 & -1 \\
1 & -3 & 0 \\
-1 & 1 & 2
\end{bmatrix} = 
\begin{bmatrix}
4 & 4 & -2 \\
3 & -1 & -3 \\
2 & 5 & 2
\end{bmatrix}
$$

Note that matrices of different sizes can NOT be added.

Matrix Multiplication

**Definition:** Let $A$ be an $m \times k$ matrix and $B$ be a $k \times n$ matrix. The product of $A$ and $B$, denoted by $AB$, is the $m \times n$ matrix that has its $(i,j)$th element equal to the sum of the products of the corresponding elements from the $i$th row of $A$ and the $j$th column of $B$. In other words, if $AB = [c_{ij}]$ then $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ik}b_{kj}$.

**Example:**

$$
\begin{bmatrix}
1 & 0 & 4 \\
2 & 1 & 1 \\
3 & 1 & 0 \\
0 & 2 & 2
\end{bmatrix} \times 
\begin{bmatrix}
2 & 4 \\
1 & 1 \\
3 & 0 \\
2 & 2
\end{bmatrix} = 
\begin{bmatrix}
14 & 4 \\
8 & 9 \\
7 & 13 \\
8 & 2
\end{bmatrix}
$$

Undefined when number of columns in the first matrix is not the same as number of rows in the second.

Illustration of Matrix Multiplication

- The Product of $A = [a_{ij}]$ and $B = [b_{ij}]$

$$
A = 
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1k} \\
a_{21} & a_{22} & \ldots & a_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mk}
\end{bmatrix}
B = 
\begin{bmatrix}
b_{11} & a_{12} & \ldots & b_{1j} & \ldots & b_{1n} \\
b_{21} & a_{22} & \ldots & b_{2j} & \ldots & b_{2n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
b_{k1} & a_{k2} & \ldots & b_{kj} & \ldots & b_{kn}
\end{bmatrix}
AB = 
\begin{bmatrix}
c_{11} & c_{12} & \ldots & c_{1j} & \ldots & c_{1n} \\
c_{21} & c_{22} & \ldots & c_{2j} & \ldots & c_{2n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
c_{m1} & c_{m2} & \ldots & c_{mj} & \ldots & c_{mn}
\end{bmatrix}
$$

$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ik}b_{kj}$

Example use of Matrix Multiplication

- Solving set of linear equations

$$J + M + S = 100 \quad \rightarrow \quad 1J + 1M + 1S = 100$$
$$-J + S = 40 \quad \rightarrow \quad -1J + 0M + 1S = 40$$
$$J + M - S = 0 \quad \rightarrow \quad 1J + 1M - 1S = 0$$

$$
\begin{bmatrix}
1 & 1 & 1 \\
-1 & 0 & 1 \\
1 & 1 & -1
\end{bmatrix} \times 
\begin{bmatrix}
J \\
M \\
S
\end{bmatrix} = 
\begin{bmatrix}
100 \\
40 \\
0
\end{bmatrix}
$$
Is Matrix Multiplication Commutative

**Example:** Let \( A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \)

Does \( AB = BA \)?

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Identity Matrix and Powers of Matrices

**Definition:** The *identity matrix of order* \( n \) is the \( m \times n \) matrix \( I_n = [\delta_{ij}] \), where \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) if \( i \neq j \).

\[
I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}
\]

\( AI_n = I_m A = A \)

when \( A \) is an \( m \times n \) matrix

Powers of square matrices can be defined. When \( A \) is an \( n \times n \) matrix, we have:

\[ A^0 = I_n \quad A^r = A A A \cdots A \]

\( r \) times

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Transposes of Matrices

**Definition:** Let \( A = [a_{ij}] \) be an \( m \times n \) matrix. The *transpose of* \( A \), denoted by \( A^t \), is the \( n \times m \) matrix obtained by interchanging the rows and columns of \( A \).

If \( A^t = [b_{ij}] \), then \( b_{ij} = a_{ji} \) for \( i = 1, 2, \ldots, n \)

and \( j = 1, 2, \ldots, m \).

The transpose of the matrix \( \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \) is the matrix \( \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \)