Announcements

- Read for next time Chap. 3.1-3.3
- No recitation Oct 11 or 14

Interview Time ….

We will solve a Microsoft interview question

Problem

Four guys want to cross a bridge that can only hold two people at one time. It is pitch dark and they only have one flashlight, so people must cross either alone or in pairs (bringing the flashlight). Their walking speeds allow them to cross in 1, 2, 5, and 10 minutes, respectively. Is it possible for them to all cross in 17 minutes?
Get The Problem Right!

• Given any context you should double check that you read/heard it correctly!

• You should be able to repeat the problem back to the source and have them agree that you understand the issue

Intuitive, But False

• “$10 + 1 + 5 + 1 + 2 = 19$, so the four guys just can’t cross in 17 minutes”

• “Even if the fastest guy is the one to shuttle the others back and forth – you use at least $10 + 1 + 5 + 1 + 2 > 17$ minutes”

Problem

Four guys want to cross a bridge that can only hold two people at one time. It is pitch dark and they only have one flashlight, so people must cross either alone or in pairs (bringing the flashlight). Their walking speeds allow them to cross in 1, 2, 5, and 10 minutes, respectively. Is it possible for them to all cross in 17 minutes?

Vocabularize Self-Proofing

• As you talk to yourself, make sure to tag assertions with phrases that denote degrees of conviction
Keep track of what you actually know – remember what you merely suspect

• “10 + 1 + 5 + 1 + 2 = 19, so it would be weird if the four guys could cross in 17 minutes”

• “even if we use the fastest guy to shuttle the others, they take too long.”

• If it is possible, there must be more than one guy doing the return trips: it must be that someone gets deposited on one side and comes back for the return trip later!

• Suppose we leave 1 for a return trip later

• We start with 1 and X and then X returns

  Total time: 2X

  Thus, we start with 1, 2 go over and 2 comes back…. 

  • 1 2 5
  10
5 and 10
“Load Balancing”:

- Handle our hardest work loads in parallel! Work backwards by assuming 5 and 10 walk together
Words To The Wise

• Keep It Simple

Don’t Fool Yourself

Chap. 3.1 Algorithms

**Definition:** An *algorithm* is a finite set of precise instructions for performing a computation or for solving a problem.

**Example:** Describe an algorithm for finding the maximum value in a finite sequence of integers.

**Solution:** Perform the following steps:

1. Set the temporary maximum equal to the first integer in the sequence.
2. Compare the next integer in the sequence to the temporary maximum.
   - If it is larger than the temporary maximum, set the temporary maximum equal to this integer.
3. Repeat the previous step if there are more integers. If not, stop.
4. When the algorithm terminates, the temporary maximum is the largest integer in the sequence.

Specifying Algorithms

- Algorithms can be specified in English or in *pseudocode*.
- Pseudocode is an intermediate step between an English and coding using a programming language.
- Appendix 3 specifies pseudocode for this book (similar to Java)
- Pseudocode helps analyze the time required to solve a problem using an algorithm, independent of the actual programming language used to implement algorithm.
Properties of Algorithms

- **Input**: An algorithm has input values from a specified set.
- **Output**: From the input values, the algorithm produces the output values from a specified set. The output values are the solution.
- **Correctness**: An algorithm should produce the correct output values for each set of input values.
- **Finiteness**: An algorithm should produce the output after a finite number of steps for any input.
- **Effectiveness**: It must be possible to perform each step of the algorithm correctly and in a finite amount of time.
- **Generality**: The algorithm should work for all problems of the desired form.

Finding the Maximum Element in a Finite Sequence

- The algorithm in pseudocode:

```plaintext
procedure max(a_1, a_2, ..., a_n; integers)
    max := a_1
    for i := 2 to n
        if max < a_i then max := a_i
    return max \{max is the largest element\}
```

- Does this algorithm have all the properties listed on the previous slide?

Problem

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- A function f from A to B is “onto” iff for every element b there is an element a with f(a) = b.
Solution: Algorithm

- Assume A has n elements, B has m elements.

- Keep a count for each element in B, setting counts to 0.

- For each \( a \in A \) compute \( f(a) = b \) and add one to b’s count.

- If any b has a count of 0, then “not onto”, otherwise “onto”

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Some Example Algorithm Problems

- Three classes of problems will look at in this chapter

  1. *Searching Problems*: finding the position of a particular element in a list.
  2. *Sorting problems*: putting the elements of a list into increasing order.
  3. *Optimization Problems*: determining the optimal value (maximum or minimum) of a particular quantity over all possible inputs.

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Greedy Algorithms

- *Optimization problems* minimize or maximize some parameter over all possible inputs.

- Examples:
  - Finding a route between two cities with the smallest total mileage.
  - Determining how to encode messages using the fewest possible bits.

- SOLVED using a greedy algorithm, which makes the “best” choice at each step. Making the “best choice” at each step does not necessarily produce an optimal solution to the overall problem, but in many instances, it does.

- Try to prove that this approach always produces an optimal solution, or find a counterexample to show that it does not.
Greedy Algorithms: Making Change

Example: Design a greedy algorithm for making change (in U.S. money) of \( n \) cents with the following coins: quarters (25 cents), dimes (10 cents), nickels (5 cents), and pennies (1 cent), using the least total number of coins.

Idea: At each step choose the coin with the largest possible value that does not exceed the amount of change left.

1. If \( n = 67 \) cents, first choose a quarter leaving \( 67 - 25 = 42 \) cents. Then choose another quarter leaving \( 42 - 25 = 17 \) cents.
2. Then choose 1 dime, leaving \( 17 - 10 = 7 \) cents.
3. Choose 1 nickel, leaving \( 7 - 5 = 2 \) cents.
4. Choose a penny, leaving one cent. Choose another penny leaving 0 cents.

Greedy Change-Making Algorithm

Solution: Greedy change-making algorithm for \( n \) cents. The algorithm works with any coin denominations \( c_1, c_2, \ldots, c_r \).

\[
\text{procedure } \text{change}(c_1, c_2, \ldots, c_r; \text{ values of coins, where } c_1 > c_2 > \ldots > c_r; \text{ } n: \text{ a positive integer})
\]

\[
\text{for } i := 1 \text{ to } r \\
\text{ } d_i := 0 \text{ [ } d_i \text{ counts the coins of denomination } c_i \text{]}
\]

\[
\text{while } n \geq c_i \\
\text{ } d_i := d_i + 1 \text{ [add a coin of denomination } c_i \text{]}
\]

\[
\text{ } n = n - c_i \\
[d_i \text{ counts the coins } c_i]
\]

– For the example of U.S. currency, we may have quarters, dimes, nickels and pennies, with \( c_1 = 25, c_2 = 10, c_3 = 5, \) and \( c_4 = 1 \).

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What is the brute-force way to show Optimality for U.S. Coins?

• Try every possible solution!
• List every possible combination of coins to show the greedy method worked.
  – That shows it worked for one case
• But does it work for every possible amount?

Proving Optimality for U.S. Coins

• Show that the change making algorithm for U.S. coins is optimal.
  Lemma 1: If \( n \) is a positive integer, then \( n \) cents in change using quarters, dimes, nickels, and pennies, using the fewest coins possible has at most 2 dimes, 1 nickel, 4 pennies, and cannot have 2 dimes and a nickel. The total amount of change in dimes, nickels, and pennies must not exceed 24 cents.
  Proof: By contradiction

Theorem: The greedy change-making algorithm for U.S. coins produces change using the fewest coins possible.
Proof: By contradiction.
Proving Optimality for U.S. Coins

**Theorem:** The greedy change-making algorithm for U.S. coins produces change using the fewest coins possible.

**Proof:** By contradiction.
1. Assume there is a positive integer $n$ such that change can be made for $n$ cents using quarters, dimes, nickels, and pennies, with a fewer total number of coins than given by the algorithm.
2. Then, $q' \leq q$ where $q'$ is the number of quarters used in this optimal way and $q$ is the number of quarters in the greedy algorithm’s solution. But this is not possible by Lemma 1, since the value of the coins other than quarters can not be greater than 24 cents.
3. Similarly, by Lemma 1, the two algorithms must have the same number of dimes, nickels, and pennies.

Greedy Change-Making Algorithm

- Optimality depends on the denominations available.
- For U.S. coins, optimality still holds if we add half dollar coins (50 cents) and dollar coins (100 cents).
- But if we allow only quarters (25 cents), dimes (10 cents), and pennies (1 cent), the algorithm no longer produces the minimum number of coins.
  - Give an example amount that it doesn’t work for.
  - Consider the example of 31 cents.
    - The optimal number of coins is 4 - 3 dimes and 1 penny.
    - Greedy algorithm is 7 coins – 1 quarter and 6 pennies.

Scheduling problems are huge!

- How did Duke schedule the rooms assigned to courses for this semester?
- How do they schedule students to take courses?
Scheduling problems are huge!

• How did Duke schedule the rooms assigned to courses for this semester?
  – Dept has half at 10-2
• How do they schedule students to take courses?
  – Students do it, FIFO

Greedy Scheduling

Example: We have a group of proposed talks with start and end times. Construct a greedy algorithm to schedule as many as possible in a lecture hall, under the following assumptions:
  – When a talk starts, it continues till the end.
  – No two talks can occur at the same time.
  – A talk can begin at the same time that another ends.
  – Once we have selected some of the talks, we cannot add a talk which is incompatible with those already selected because it overlaps at least one of these previously selected talks.

What algorithm should we use to pick talks?

• The talk that starts earliest among those compatible with already chosen talks?
• The talk that is shortest among those already compatible?
• The talk that ends earliest among those compatible with already chosen talks?

Greedy Scheduling

• Picking the shortest talk doesn’t work.

But picking the one that ends soonest does work. The algorithm is specified on the next page.
Greedy Scheduling algorithm

Solution: At each step, choose the talks with the earliest ending time among the talks compatible with those selected.

procedure schedule($s_1 \leq s_2 \leq \ldots \leq s_n$: start times, $e_1 \leq e_2 \leq \ldots \leq e_n$: end times)
sort talks by finish time and reorder so that $e_1 \leq e_2 \leq \ldots \leq e_n$
$S := \emptyset$
for $j := 1$ to $n$
    if talk $j$ is compatible with $S$
        $S := S \cup \{\text{talk } j\}$
return $S$ [ $S$ is the set of talks scheduled]

Halting Problem

• One of the most famous computer science problems
• Can you decide if a program you wrote will ever halt on an input?

Halting Problem

Example: Can we develop a procedure that takes as input a computer program along with its input and determines whether the program will eventually halt with that input.

• Solution: Proof by contradiction.
• Assume that there is such a procedure and call it $H(P,I)$. The procedure $H(P,I)$ takes as input a program $P$ and the input $I$ to $P$.
  – $H$ outputs “halt” if it is the case that $P$ will stop when run with input $I$.
  – Otherwise, $H$ outputs “loops forever.”

Halting Problem

• Since a program is a string of characters, we can call $H(P,P)$. Construct a procedure $K(P)$, which works as follows.
  – If $H(P,P)$ outputs “loops forever” then $K(P)$ halts.
  – If $H(P,P)$ outputs “halt” then $K(P)$ goes into an infinite loop printing “ha” on each iteration.
Halting Problem

• Now we call $K$ with $K$ as input, i.e. $K(K)$.
  – If the output of $H(K,K)$ is “loops forever” then $K(K)$ halts. A Contradiction.
  – If the output of $H(K,K)$ is “halts” then $K(K)$ loops forever. A Contradiction.
• Therefore, there cannot be a procedure that can decide whether or not an arbitrary program halts. The halting problem is unsolvable.

Xkcd version of the Halting problem

```c
#define DOES IT HALT (PROGRAM) {
    return true;
}
```

The big picture solution to the halting problem