Strong Induction

- **Strong Induction**: To prove that $P(n)$ is true for all positive integers $n$, where $P(n)$ is a propositional function, complete two steps:
  - **Basis Step**: Verify that the proposition $P(1)$ is true.
  - **Inductive Step**: Show the conditional statement $[P(1) \land P(2) \land \cdots \land P(k)] \rightarrow P(k + 1)$ holds for all positive integers $k$.

Strong Induction is sometimes called the second principle of mathematical induction or complete induction.

Strong Induction and the Infinite Ladder

Strong induction tells us that we can reach all rungs if:
1. We can reach the first rung of the ladder.
2. For every integer $k$, if we can reach the first $k$ rungs, then we can reach the $(k + 1)$st rung.

To conclude that we can reach every rung by strong induction:
- **BASIS STEP**: $P(1)$ holds
- **INDUCTIVE STEP**: Assume $P(1) \land P(2) \land \cdots \land P(k)$ holds for an arbitrary integer $k$ and show that $P(k + 1)$ must also hold.

We will have then shown by strong induction that for every positive integer $n$, $P(n)$ holds, i.e., we can reach the $n$th rung of the ladder.
Proof using Strong Induction

Example: Suppose we can reach the first and second rungs of an infinite ladder, and we know that if we can reach a rung, then we can reach two rungs higher. Prove that we can reach every rung.

(Try this with mathematical induction.)

Solution: Prove the result using strong induction.
– BASIS STEP: We can reach the first step.
– INDUCTION STEP: The inductive hypothesis is that we can reach the first \( k \) rungs, for any \( k \geq 2 \).
– We can reach the \((k+1)\)st rung since we can reach the \((k-1)\)st rung by the inductive hypothesis.
– Hence, we can reach all rungs of the ladder.

Which Form of Induction Should Be Used?

• Can always use strong induction instead of mathematical induction.
• (if it is simpler use mathematical induction).
• The principles of mathematical induction, strong induction, and the well-ordering property are all equivalent.
• Sometimes it is clear how to proceed using one of the three methods, but not the other two.

Completion of the proof of the Fundamental Theorem of Arithmetic

Example: Show that if \( n \) is an integer greater than 1, then \( n \) can be written as the product of primes.

Solution: Let \( P(n) \) be the proposition that \( n \) can be written as a product of primes.
– BASIS STEP: \( P(2) \) is true since 2 itself is prime.
– INDUCTION STEP: The inductive hypothesis is \( P(j) \) is true for all integers \( j \) with \( 2 \leq j \leq k \).
– To show that \( P(k + 1) \) must be true under this assumption, two cases need to be considered:
Example: Show that if \( n \) is an integer greater than 1, then \( n \) can be written as the product of primes.

Solution: Let \( P(n) \) be the proposition that \( n \) can be written as a product of primes.

- BASIS STEP: \( P(2) \) is true since 2 itself is prime.
- INDUCTIVE STEP: The inductive hypothesis is \( P(j) \) is true for all integers \( j \) with \( 2 \leq j \leq k \).
- To show that \( P(k + 1) \) must be true under this assumption, two cases need to be considered:
  - If \( k + 1 \) is prime, then \( P(k + 1) \) is true.
  - Otherwise, \( k + 1 \) is composite and can be written as the product of two positive integers \( a \) and \( b \) with \( 2 \leq a \leq b < k + 1 \). By the inductive hypothesis, \( a \) and \( b \) can be written as the product of primes and therefore \( k + 1 \) can also be written as the product of those primes.

Hence, it has been shown that every integer greater than 1 can be written as the product of primes.

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Proof using Strong Induction

Example: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: Let \( P(n) \) be the proposition that postage of \( n \) cents can be formed using 4-cent and 5-cent stamps.

- BASIS STEP: \( P(12), P(13), P(14), \) and \( P(15) \) hold.
  - \( P(12) \) uses three 4-cent stamps.
  - \( P(13) \) uses two 4-cent stamps and one 5-cent stamp.
  - \( P(14) \) uses one 4-cent stamp and two 5-cent stamps.
  - \( P(15) \) uses three 5-cent stamps.
- INDUCTIVE STEP: The inductive hypothesis states that \( P(j) \) holds for \( 12 \leq j \leq k \), where \( k \geq 15 \). Assuming the inductive hypothesis, it can be shown that \( P(k + 1) \) holds.
- Using the inductive hypothesis, to form postage of \( k + 1 \) cents, \( P(k + 1) \) holds since \( k + 1 \geq 12 \).

Hence, \( P(n) \) holds for all \( n \geq 12 \).
Proof of Same Example using Mathematical Induction

**Example:** Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

**Solution:** Let $P(n)$ be the proposition that postage of $n$ cents can be formed using 4-cent and 5-cent stamps.
- **BASIS STEP:** Postage of 12 cents can be formed using three 4-cent stamps.
- **INDUCTIVE STEP:** The inductive hypothesis $P(k)$ for any positive integer $k$ is that postage of $k$ cents can be formed using 4-cent and 5-cent stamps. To show $P(k + 1)$ where $k \geq 12$, we consider two cases:
  - If at least one 4-cent stamp has been used, then a 4-cent stamp can be replaced with a 5-cent stamp to yield a total of $k + 1$ cents.
  - Otherwise, no 4-cent stamp have been used and at least three 5-cent stamps were used. Three 5-cent stamps can be replaced by four 4-cent stamps to yield a total of $k + 1$ cents.

Hence, $P(n)$ holds for all $n \geq 12$.

Well-Ordering Property

- **Well-ordering property:** Every nonempty set of nonnegative integers has a least element.
- The well-ordering property is one of the axioms of the positive integers listed in Appendix 1.
- The well-ordering property can be used directly in proofs, as the next example illustrates.
- The well-ordering property can be generalized.
  - **Definition:** A set is well ordered if every subset has a least element.
    - $\mathbb{N}$ is well ordered under $\leq$.
    - The set of finite strings over an alphabet using lexicographic ordering is well ordered.
  - We will see a generalization of induction to sets other than the integers.

Sec 5.3 - Recursively Defined Functions

**Definition:** A recursive or inductive definition of a function consists of two steps.
- **BASIS STEP:** Specify the value of the function at zero.
- **RECURSIVE STEP:** Give a rule for finding its value at an integer from its values at smaller integers.

• A function $f(n)$ is the same as a sequence $a_0$, $a_1$, ..., where $a_i$, where $f(i) = a_i$. This was done using recurrence relations in Section 2.4.
Recursively Defined Functions

Example: Suppose $f$ is defined by:
\[ f(0) = 3, \]
\[ f(n + 1) = 2f(n) + 3 \]
Find $f(1), f(2), f(3), f(4)$

Solution:
- $f(1) =$
- $f(2) =$
- $f(3) =$
- $f(4) =$

Example: Give a recursive definition of the factorial function $n!$:

Solution:

Recursively Defined Functions

Example: Give a recursive definition of:

Solution: The first part of the definition is

The second part is
Recursively Defined Functions

**Example:** Give a recursive definition of:

\[ \sum_{k=0}^{n} a_k. \]

**Solution:** The first part of the definition is

\[ \sum_{k=0}^{0} a_k = a_0. \]

The second part is

\[ \sum_{k=0}^{n+1} a_k = \left( \sum_{k=0}^{n} a_k \right) + a_{n+1}. \]

---

Fibonacci Numbers

**Example:** The Fibonacci numbers are defined as follows:

\[ f_0 = 0 \]
\[ f_1 = 1 \]
\[ f_n = f_{n-1} + f_{n-2} \]

Find \( f_2, f_3, f_4, f_5 \).

- \( f_2 = f_1 + f_0 = 1 + 0 = 1 \)
- \( f_3 = f_2 + f_1 = 1 + 1 = 2 \)
- \( f_4 = f_3 + f_2 = 2 + 1 = 3 \)
- \( f_5 = f_4 + f_3 = 3 + 2 = 5 \)

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Lamé’s Theorem

**Lamé’s Theorem:** Let \( a \) and \( b \) be positive integers with \( a \geq b \). Then the number of divisions used by the Euclidian algorithm to find \( \gcd(a,b) \) is less than or equal to five times the number of decimal digits in \( b \).

**Proof:** in book.

- As a consequence of Lamé’s Theorem, \( O(\log b) \) divisions are used by the Euclidian algorithm to find \( \gcd(a,b) \) whenever \( a > b \).

---

Recursively Defined Sets and Structures

**Recursive definitions** of sets have two parts:
- The **basis step** specifies an initial collection of elements.
- The **recursive step** gives the rules for forming new elements in the set from those already known to be in the set.

- Sometimes the recursive definition has an exclusion rule, which specifies that the set contains nothing other than those elements specified in the basis step and generated by applications of the rules in the recursive step.
- We will always assume that the exclusion rule holds, even if it is not explicitly mentioned.
- We will later develop a form of induction, called structural induction, to prove results about recursively defined sets.
Recursively Defined Sets and Structures

Example: Set of positive integers $S$ that are multiples of 3
BASIS STEP: $3 \in S$.
RECURSIVE STEP: If $x \in S$ and $y \in S$, then $x + y$ is in $S$.

Example: The natural numbers $N$.
BASIS STEP: $0 \in N$.
RECURSIVE STEP: If $n \in N$, then $n + 1$ is in $N$.

Strings

Definition: The set $\Sigma^*$ of strings over the alphabet $\Sigma$:
BASIS STEP: $\lambda \in \Sigma^*$ ($\lambda$ is the empty string)
RECURSIVE STEP: If $w \in \Sigma^*$ and $x \in \Sigma$, then $wx \in \Sigma^*$.

Example: If $\Sigma = \{0,1\}$, the strings in $\Sigma^*$ are the set of all bit strings:

Example: If $\Sigma = \{a,b\}$, show that $aab$ is in $\Sigma^*$.
String Concatenation

**Definition:** Two strings can be combined via the operation of *concatenation*. Let $\Sigma$ be a set of symbols and $\Sigma^*$ be the set of strings formed from the symbols in $\Sigma$. We can define the concatenation of two strings, denoted by $\cdot$, recursively as follows.

**BASIS STEP:** If $w \in \Sigma^*$, then $w \cdot \lambda = w$.

**RECURSIVE STEP:** If $w_1 \in \Sigma^*$ and $w_2 \in \Sigma^*$ and $x \in \Sigma$, then $w_1 \cdot (w_2 x) = (w_1 \cdot w_2) x$.

- Often $w_1 \cdot w_2$ is written as $w_1 w_2$.
- If $w_1 = abra$ and $w_2 = cadabra$, the concatenation is $w_1 w_2 = abracadabra$.

Length of a String

**Example:** Give a recursive definition of $l(w)$, the length of the string $w$.

**Solution:** The length of a string can be recursively defined by:

- $l(\lambda) = 0$
- $l(wx) = l(w) + 1$ if $w \in \Sigma^*$ and $x \in \Sigma$.
Balanced Parentheses

Example: Give a recursive definition of the set of balanced parentheses $P$.

Solution:

BASIS STEP: $( ) \in P$

RECURSIVE STEP: If $w \in P$, then

- Show that $(( ) ( ))$ is in $P$.
- Why is $))(()$ not in $P$?

Well-Formed Formulae in Propositional Logic

Definition: The set of well-formed formulae in propositional logic involving $T$, $F$, propositional variables, and operators from the set \{$\neg,\wedge,\vee,\to,\leftrightarrow$\}.

BASIS STEP: $T,F,$ and $s$, where $s$ is a propositional variable, are well-formed formulae.

RECURSIVE STEP: If $E$ and $F$ are well formed formulae, then

are well-formed formulae.

Examples: $(p \lor q) \to (q \land F)$ is a well-formed formula.

$pq \land$ is not a well-formed formula.
Rooted Trees

**Definition:** The set of *rooted trees*, where a rooted tree consists of a set of vertices containing a distinguished vertex called the *root*, and edges connecting these vertices, can be defined recursively by these steps:

**BASIS STEP:** A single vertex $r$ is a rooted tree.

**RECURSIVE STEP:** Suppose that $T_1, T_2, \ldots, T_n$ are disjoint rooted trees with roots $r_1, r_2, \ldots, r_n$, respectively. Then the graph formed by starting with a root $r$, which is not in any of the rooted trees $T_1, T_2, \ldots, T_n$, and adding an edge from $r$ to each of the vertices $r_1, r_2, \ldots, r_n$, is also a rooted tree.

Building Up Rooted Trees

- Trees are studied extensively in Chapter 11.
- Next we look at a special type of tree, the full binary tree.

How do you construct this rooted tree recursively?

Full Binary Trees

**Definition:** The set of *full binary trees* can be defined recursively by these steps.

**BASIS STEP:** There is a full binary tree consisting of only a single vertex $r$.

**RECURSIVE STEP:** If $T_1$ and $T_2$ are disjoint full binary trees, there is a full binary tree, denoted by $T_1; T_2$, consisting of a root $r$ together with edges connecting the root to each of the roots of the left subtree $T_1$ and the right subtree $T_2$. 
Is this a full binary tree?

Building Up Full Binary Trees

What can you say about the nonleaf nodes in a full binary tree?

Induction and Recursively Defined Sets

Example: Show that the set S defined by specifying that 3 ∈ S and that if x ∈ S and y ∈ S, then x + y is in S, is the set of all positive integers that are multiples of 3.

Solution: Let A be the set of all positive integers divisible by 3. To prove that A = S, show that A is a subset of S and S is a subset of A.

- A ⊆ S: Let P(n) be the statement that 3n belongs to S.
  BASIS STEP: Assume P(k) is true.
  By the second part of the recursive definition,
  Hence, P(k + 1) is true.

- S ⊆ A:
  BASIS STEP: by the first part of recursive definition, and
  INDUCTIVE STEP: The second part of the recursive definition

By part (i) of Theorem 1 of Section 4.1, it follows that

Induction and Recursively Defined Sets

Example: Show that the set S defined by specifying that 3 ∈ S and that if x ∈ S and y ∈ S, then x + y is in S, is the set of all positive integers that are multiples of 3.

Solution: Let A be the set of all positive integers divisible by 3. To prove that A = S, show that A is a subset of S and S is a subset of A.

- A ⊆ S: Let P(n) be the statement that 3n belongs to S.
  BASIS STEP: 3 · 1 = 3 ∈ S, by the first part of recursive definition.
  INDUCTIVE STEP: Assume P(k) is true.
  By the second part of the recursive definition, if 3k ∈ S, then since 3 ∈ S, 3k + 3 = 3(k + 1) ∈ S.
  Hence, P(k + 1) is true.

- S ⊆ A:
  BASIS STEP: 3 ∈ S by the first part of recursive definition, and 3 = 3 · 1.
  INDUCTIVE STEP: The second part of the recursive definition adds x + y to S, if both x and y are in S.
  If x and y are both in A, then both x and y are divisible by 3.
  By part (i) of Theorem 1 of Section 4.1, it follows that x + y is divisible by 3.
Structural Induction

We used mathematical induction to prove a result about a recursively defined set. Next we study a more direct form induction for proving results about recursively defined sets.

Definition: To prove a property of the elements of a recursively defined set, we use structural induction.

Basis step: Show that the result holds for all elements specified in the basis step of the recursive definition.

Recursive step: Show that if the statement is true for each of the elements used to construct new elements in the recursive step of the definition, the result holds for these new elements.

• The number of vertices \( n(T) \) of a full binary tree \( T \) consisting of only a root is \( n(T) = 1 \).

• The number of vertices \( n(T) \) of a full binary tree \( T \) satisfies the following recursive formula:

\[ n(T) = 1 + n(T_1) + n(T_2). \]

Full Binary Trees

Definition: The height \( h(T) \) of a full binary tree \( T \) is defined recursively as follows:

- Basis step: The height of a full binary tree \( T \) consisting of only a root \( r \) is \( h(T) = 0 \).
- Recursive step: If \( T_1 \) and \( T_2 \) are full binary trees, then the full binary tree \( T = T_1 \cdot T_2 \) has height \( h(T) = 1 + \max(h(T_1),h(T_2)) \).

• The number of vertices \( n(T) \) of a full binary tree \( T \) satisfies the following recursive formula:

\[ n(T) = 1 + n(T_1) + n(T_2). \]
Structural Induction and Binary Trees

**Theorem:** If $T$ is a full binary tree, then $n(T) \leq 2^{h(T)+1} - 1$.

**Proof:** Use structural induction.
- **Basis Step:** The result holds for a full binary tree consisting only of a root, $n(T) = 1$ and $h(T) = 0$.
  Hence, $n(T) = 2^{0+1} - 1 = 1$.
- **Recursive Step:** Assume $n(T_1) \leq 2^{h(T_1)+1} - 1$ and also $n(T_2) \leq 2^{h(T_2)+1} - 1$ whenever $T_1$ and $T_2$ are full binary trees.

\[
\begin{align*}
n(T) &= 1 + n(T_1) + n(T_2) \quad \text{(by recursive formula of } n(T)) \\
&\leq 1 + (2^{h(T_1)+1} - 1) + (2^{h(T_2)+1} - 1) \quad \text{(by inductive hypothesis)} \\
&= 2 \max(2^{h(T_1)+1}, 2^{h(T_2)+1}) - 1 \\
&= 2 \cdot 2 \max(h(T_1), h(T_2)) + 1 - 1 \\
&= 2 \cdot 2^{h(T)} - 1 \\
&= 2^{h(T)+1} - 1 \quad \text{(by recursive definition of } h(T))
\end{align*}
\]

Generalized Induction

- **Generalized induction** is used to prove results about sets other than the integers that have the well-ordering property. (*explored in more detail in Chapter 9*)
- For example, consider an ordering on $\mathbb{N} \times \mathbb{N}$, ordered pairs of nonnegative integers. Specify that $(x_1, y_1)$ is less than or equal to $(x_2, y_2)$ if either $x_1 < x_2$, or $x_1 = x_2$ and $y_1 < y_2$. This is called the **lexicographic ordering**.
- Strings are also commonly ordered by a **lexicographic ordering**.