Announcements

- Read for next time Chap. 6.5-6.6
- Homework 6 due Tuesday
- Recitation this week – bring laptop

Counting II: Recurring Problems and Correspondences

\[(\text{Ace} + \text{Queen} + \text{King})(\text{Jack} + \text{Queen}) = ?\]
Correspondence Principle
If two finite sets can be placed into 1-1 onto correspondence, then they have the same size.

If a finite set A has a k-to-1 correspondence to finite set B, then $|B| = |A|/k$.

The number of subsets of an n-element set is $2^n$.

Sometimes it is easiest to count the number of objects with property Q, by counting the number of objects that do not have property Q.
The number of subsets of size \( r \) that can be formed from an \( n \)-element set is:

\[
\frac{n!}{r!(n-r)!} = \binom{n}{r}
\]

A choice tree provides a “choice tree representation” of a set \( S \), if

1. Each leaf label is in \( S \), and each element of \( S \) is some leaf label
2. No two leaf labels are the same

Product Rule (Rephrased)

Suppose every object of a set \( S \) can be constructed by a sequence of choices with \( P_1 \) possibilities for the first choice, \( P_2 \) for the second, and so on.

**IF**

1. Each sequence of choices constructs an object of type \( S \)

**AND**

2. No two different sequences create the same object

**THEN**

There are \( P_1P_2P_3\ldots P_n \) objects of type \( S \)

How Many Different Orderings of Deck With 52 Cards?

What object are we making? Ordering of a deck

Construct an ordering of a deck by a sequence of 52 choices:

- 52 possible choices for the first card;
- 51 possible choices for the second card;
- ...;
- 2 possible choices for the 51st card;
- 1 possible choice for the 52nd card.

By product rule: \( 52 \times 51 \times 50 \times \ldots \times 2 \times 1 = 52! \)
The Sleuth’s Criterion

There should be a unique way to create an object in S.

In other words:

For any object in S, it should be possible to reconstruct the (unique) sequence of choices which lead to it.

The three big mistakes people make in associating a choice tree with a set S are:

1. Creating objects not in S
2. Leaving out some objects from the set S
3. Creating the same object two different ways

DEFENSIVE THINKING
ask yourself:

Am I creating objects of the right type?

Can I reverse engineer my choice sequence from any given object?

Let’s use our principles to extend our reasoning to different types of objects
Counting Poker Hands

52 Card Deck, 5 card hands

4 possible suits:

♥ ♦ ♣ ♠

13 possible ranks:

2,3,4,5,6,7,8,9,10,J,Q,K,A

Pair: set of two cards of the same rank
Straight: 5 cards of consecutive rank
Flush: set of 5 cards with the same suit

Ranked Poker Hands

Straight Flush: a straight and a flush
4 of a kind: 4 cards of the same rank
Full House: 3 of one rank and 2 of another
Flush: a flush, but not a straight
Straight: a straight, but not a flush
3 of a kind: 3 of the same rank, but not a full house or 4 of a kind
2 Pair: 2 pairs, but not 4 of a kind or a full house
A Pair

Straight Flush

Choices for rank? Possible suits?

9 choices for rank of lowest card at the start of the straight
4 possible suits for the flush

9 × 4 = 36
Straight Flush
Choices for rank? Possible suits?

9 choices for rank of lowest card at
the start of the straight

4 possible suits for the flush

$9 \times 4 = 36$

$$\frac{36}{\binom{52}{5}} = \frac{36}{2,598,960} = 1 \text{ in } 72,193.333...$$

4 of a Kind
Choices of rank? Other choices?

13 choices of rank

48 choices for remaining card

$13 \times 48 = 624$

$$\frac{624}{\binom{52}{5}} = \frac{624}{2,598,960} = 1 \text{ in } 4,165$$

Flush
Choices of suit? Choices of cards?

5112 flushes

21

22

23

24
Flush
Choices of suit? Choices of cards?
4 choices of suit
\[
\binom{13}{5} \text{ choices of cards}
\]
\[
4 \times 1287 = 5148
\]
“but not a straight flush…”
- 36 straight flushes
\[
5112 \text{ flushes}
\]
\[
\frac{5112}{\binom{52}{5}} = 1 \text{ in } 508.4...
\]

Straight
Choices of lowest card? Suits?
9 choices of lowest card
\[
4^5 \text{ choices of suits for 5 cards}
\]
\[
9 \times 1024 = 9216
\]
“but not a straight flush…”
- 36 straight flushes
\[
9180 \text{ straights}
\]
\[
\frac{9180}{\binom{52}{5}} = 1 \text{ in } 283.1...
\]

Storing Poker Hands: How many bits per hand?
I want to store a 5-card poker hand using the smallest number of bits (space efficient)
Order the 2,598,560 Poker Hands Lexicographically (or in any fixed way)

To store a hand all I need is to store its index, which requires $\lceil \log_2(2,598,560) \rceil = 22$ bits

Hand 0000000000000000000000
Hand 0000000000000000000001
Hand 0000000000000000000010
. .

22 Bits is OPTIMAL

$2^{21} = 2,097,152 < 2,598,560$

Thus there are more poker hands than there are 21-bit strings

Hence, you can’t have a 21-bit string for each hand

Binary (Boolean) Choice Tree

A binary (Boolean) choice tree is a choice tree where each internal node has degree 2

Usually the choices are labeled 0 and 1

22 Bits is OPTIMAL

$2^{21} = 2,097,152 < 2,598,560$

A binary choice tree of depth 21 can have at most $2^{21}$ leaves.

Hence, there are not enough leaves for all 5-card hands.
An n-element set can be stored so that each element uses \( \lceil \log_2(n) \rceil \) bits.

Furthermore, any representation of the set will have some string of at least that length.

Information Counting Principle:
If each element of a set can be represented using \( k \) bits, the size of the set is bounded by \( 2^k \).

Information Counting Principle:
Let \( S \) be a set represented by a depth-\( k \) binary choice tree, the size of the set is bounded by \( 2^k \).

ONGOING MEDITATION:
Let \( S \) be any set and \( T \) be a binary choice tree representation of \( S \).

Think of each element of \( S \) being encoded by binary sequences of choices that lead to its leaf.

We can also start with a binary encoding of a set and make a corresponding binary choice tree.
Now, for something completely different…

How many ways to rearrange the letters in the word “SYSTEMS”? 

7 places to put the Y, 
6 places to put the T, 
5 places to put the E, 
4 places to put the M, 
and the S’s are forced

$7 \times 6 \times 5 \times 4 = 840$

Let’s pretend that the S’s are distinct: 
$S_1YS_2TEM_3$
Let’s pretend that the S’s are distinct:

$S_1YS_2TEMS_3$

There are $7!$ permutations of $S_1YS_2TEMS_3$.

But when we stop pretending, we see that we have counted each arrangement of $SYSTEMS$ 3! times, once for each of 3! rearrangements of $S_1S_2S_3$.

$$\frac{7!}{3!} = 840$$

Arrange $n$ symbols: $r_1$ of type 1, $r_2$ of type 2, …, $r_k$ of type $k$.

$$\binom{n}{r_1} \binom{n-r_1}{r_2} \cdots \binom{n-r_1-r_2-\cdots-r_{k-1}}{r_k}$$

$$= \frac{n!}{(n-r_1)!r_1!} \frac{(n-r_1)!}{(n-r_1-r_2)!r_2!} \cdots$$

$$= \frac{n!}{r_1!r_2! \cdots r_k!}$$

DUKEBLUEDEVILS

2 D’s, 2 U’s, 3 E’s, 2 L’s

$$\frac{14!}{2!2!3!2!} = 1,811,214,400$$
Remember:
The number of ways to arrange \( n \) symbols with \( r_1 \) of type 1, \( r_2 \) of type 2, \( \ldots \), \( r_k \) of type \( k \) is:

\[
\frac{n!}{r_1!r_2! \ldots r_k!}
\]

5 distinct pirates want to divide 20 identical, indivisible bars of gold. How many different ways can they divide up the loot?

Sequences with 20 G’s and 4 /’s

\[
GG/G/GGGGGGGGGGGGGGGGGG/G
\]

represents the following division among the pirates: 2, 1, 0, 17, 0

In general, the \( i \)-th pirate gets the number of G’s after the \( i-1 \)-st / and before the \( i \)-th /

This gives a correspondence between divisions of the gold and sequences with 20 G’s and 4 /’s

How many different ways to divide up the loot?

Sequences with 20 G’s and 4 /’s
How many different ways to divide up the loot?

Sequences with 20 G’s and 4 l’s

\[
\binom{24}{4}
\]

How many different ways can \( n \) distinct pirates divide \( k \) identical, indivisible bars of gold?

\[
\binom{n + k - 1}{n - 1} = \binom{n + k - 1}{k}
\]

How many integer solutions to the following equations?

\[
x_1 + x_2 + x_3 + x_4 + x_5 = 20
\]

\( x_1, x_2, x_3, x_4, x_5 \geq 0 \)

Think of \( x_k \) are being the number of gold bars that are allotted to pirate \( k \)

\[
\binom{24}{4}
\]

How many integer solutions to the following equations?

\[
x_1 + x_2 + x_3 + \ldots + x_n = k
\]

\( x_1, x_2, x_3, \ldots, x_n \geq 0 \)
How many integer solutions to the following equations?

\[ x_1 + x_2 + x_3 + \ldots + x_n = k \]
\[ x_1, x_2, x_3, \ldots, x_n \geq 0 \]

\[
\binom{n + k - 1}{n - 1} = \binom{n + k - 1}{k}
\]

Identical/Distinct Dice

Suppose that we roll seven dice

\[ \Box \Box \Box \Box \Box \Box \Box \Box \]

How many different outcomes are there, if order matters?

What if order doesn’t matter? (E.g., Yahtzee!)

Identical/Distinct Dice

How did we get that last one, when order doesn’t matter?

There are 7 Dice, order doesn’t matter:

\[ \Box \Box \Box \Box \Box \Box \Box \]

Partition the dice based on their values: suppose you have  

\[ \underline{3623225} \]

\[ \underline{Or/\Box/\Box/\Box/\Box/\Box/\Box/\Box} \]

7 dice + 5 dividers = 12

53

54

55

56
How did we get that last one, when order doesn’t matter?
There are 7 Dice, order doesn’t matter: DDDDDDD
Partition the dice based on their values: suppose you have 3623225
/22/33/5/6
Or
/DDD/DD/DD/D

\[
\binom{12}{5} = \binom{12}{7}
\]

7 dice + 5 dividers = 12

Multisets
A multiset is a set of elements, each of which has a multiplicity
The size of the multiset is the sum of the multiplicities of all the elements
Example:
\{X, Y, Z\} with m(X)=0 m(Y)=3, m(Z)=2
Unary visualization:
\{Y, Y, Y, Z, Z\}

Counting Multisets
The number of ways to choose a multiset of size k from n types of elements is:
\[
\binom{n+k-1}{n-1} = \binom{n+k-1}{k}
\]
Back to the Pirates

How many ways are there of choosing 20 pirates from a set of 5 pirates, with repetitions allowed?

Polynomials Express Choices and Outcomes

Products of Sum = Sums of Products

\[
\binom{5 + 20 - 1}{20} = \binom{24}{20} = \binom{24}{4}
\]
Polynomials Express Choices and Outcomes
Products of Sum = Sums of Products

\[(b_1 + b_2 + b_3)(t_1 + t_2) = b_1 t_1 + b_1 t_2 + b_2 t_1 + b_2 t_2 + b_3 t_1 + b_3 t_2\]
There is a correspondence between paths in a choice tree and the cross terms of the product of polynomials!

Choice Tree for Terms of $(1+X)^3$

Combine like terms to get:

$$1 + 3X + 3X^2 + X^3$$

What is a Closed Form Expression For $c_k$?

$$(1+X)^n = c_0 + c_1X + c_2X^2 + \ldots + c_nX^n$$

After multiplying things out, but before combining like terms, we get $2^n$ cross terms, each corresponding to a path in the choice tree $c_k$, the coefficient of $X^k$, is the number of paths with exactly $k$ X’s
What is a Closed Form Expression For $c_k$?

$(1+X)^n = c_0 + c_1X + c_2X^2 + \ldots + c_nX^n$

After multiplying things out, but before combining like terms, we get $2^n$ cross terms, each corresponding to a path in the choice tree $c_k$, the coefficient of $X^k$, is the number of paths with exactly $k$ X’s

$$c_k = \binom{n}{k}$$

The Binomial Formula

$(1+X)^n = \binom{n}{0}X^0 + \binom{n}{1}X^1 + \ldots + \binom{n}{n}X^n$

Coefficients?

| Coefficients? | 75 | 74
|---------------|-----------------|
| $n\choose 0$ | 4 | 0
| $n\choose 1$ | 4 | 1
| $n\choose 2$ | 4 | 2
| $n\choose 3$ | 4 | 3
| $n\choose 4$ | 4 | 4

The Binomial Formula

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

Coefficients?

| Coefficients? | 75 | 74
|---------------|-----------------|
| $n\choose 0$ | 4 | 0
| $n\choose 1$ | 4 | 1
| $n\choose 2$ | 4 | 2
| $n\choose 3$ | 4 | 3
| $n\choose 4$ | 4 | 4
The Binomial Formula

\[(X+Y)^n = \sum_{k=0}^{n} \binom{n}{k} X^{n-k} Y^k\]

\[\binom{n}{0} X^n Y^0 + \binom{n}{1} X^{n-1} Y^1 + \ldots + \binom{n}{n} X^0 Y^n\]

What is the coefficient of EMSTY in the expansion of \((E + M + S + T + Y)^5\)?

5!
What is the coefficient of \( \text{EMS}^3 \text{TY} \) in the expansion of \((E + M + S + T + Y)^7\)?

The number of ways to rearrange the letters in the word SYSTEMS is \( \frac{7!}{3!} \).

What is the coefficient of \( \text{BA}^3 \text{N}^2 \) in the expansion of \((B + A + N)^6\)?

The number of ways to rearrange the letters in the word BANANA is \( \frac{6!}{3! \cdot 2!} \).
What is the coefficient of \((X_1^{r_1}X_2^{r_2}\ldots X_k^{r_k})\) in the expansion of \((X_1+X_2+X_3+\ldots+X_k)^n\)?

\[
\frac{n!}{r_1!r_2!\ldots r_k!}
\]

There is much, much more to be said about how polynomials encode counting questions!

References
Applied Combinatorics, by Alan Tucker