Announcements

- Reading this week Chap. 8.1-8.3, Chap10.1-10.4, 10.7-10.8, Chapter 11.1
  - Looks like a lot, but also a lot of review from CompSci 201, plus some new items
- Last Recitation on Friday (optional)
- Test back today

Recurrence Relations

- model lots of problems

  - The Tower of Hanoi
  - Divide and conquer algorithms
    - Sorting algorithm mergesort
    - Sorting algorithm quicksort
  - Tree algorithms
    - Searching for an element in a binary search tree
    - Listing out all elements in a binary search tree

Solving a recurrence relation

- Problem sets up as a recurrence
  - Must have a base case
- Solve the recurrence
  - Use substitution
- Prove correctness
  - Proof by induction
Example 1

- \( a_n = a_{n-1} + c \)
- \( a_0 = 1 \)

- Solve recurrence
- Then prove true by induction
- What is this an example of?

Proof by induction

Basis: \( a_0 = 1 \)

\[
\begin{align*}
a_n &= a_{n-1} + c \\
&= [a_{n-2} + c] + c \quad \text{substitute } a_{n-1} = a_{n-2} + c \\
&= a_{n-2} + 2c \\
&= [a_{n-3} + c] + 2c \quad \text{substitute } a_{n-2} = a_{n-3} + c \\
&= a_{n-3} + 3c \\
& \vdots \\
&= a_{n-k} + kc \\
& \vdots \\
&= a_0 + nc = cn + 1
\end{align*}
\]

Assume: for all \( k < n \)

\[
\begin{align*}
a_n &= a_{n-1} + c \quad \text{Show true for } k=n \\
&= \\
&= \\
&=
\end{align*}
\]
Proof by induction

Basis: \( a_0 = 1 \)
\[ a_n = c_n + 1 = c(0) + 1 = 1 \checkmark \]

Assume: \( a_k = ck + 1 \) for all \( k < n \)
\[ a_n = a_{n-1} + c \quad \text{Show true for } k = n \]
\[ = [c(n-1) + 1] + c \quad \text{by I. H.} \]
\[ = cn - c + 1 + c \]
\[ = cn + 1 \checkmark \]

Worst case binary search tree

Example 2

• \( a_n = 2a_{n-1} + c \)
• \( a_0 = 0 \)

• Solve recurrence
• Then prove true by induction
• What is this an example of?
Example 2

\[ a_0 = 0 \]
\[ a_n = 2a_{n-1} + c \quad \rightarrow \text{Means} \]
\[ = k \quad a_{n-k} + c(2^{k-1} + \ldots + 2 + 1) \]
\[ = 2^k a_{n-k} + c(2^k - 1) \quad \text{Substituting for sum} \]

\[ \rightarrow \text{Means} \quad a_{n-1} = 2a_{n-2} + c \]
\[ = 2 \quad [2a_{n-2} + c] + c \quad \text{Substituting for } a_{n-2} \]
\[ = 2^2 \quad a_{n-2} + 2c + c \]
\[ = 2^2 \quad [2 \cdot a_{n-3} + c] + 2c + c \quad \text{Substituting for } a_{n-3} \]
\[ = 2^3 \quad a_{n-3} + 4c + 2c + c \]
\[ \ldots \]

Note: this is exponential!
Proof by Induction

Basis: \( a_0 = 0 \)

Assume: \( a_k = \) for all \( k < n \)

\[
a_n = 2a_{n-1} + c
\]

\[
= c(2^n - 1)
\]

Proof by Induction

Basis: \( a_0 = 0 \)

\[
a_0 = c(2^0 - 1) = c(1 - 1) = 0
\]

Assume: \( a_k = c(2^k - 1) \) for all \( k < n \)

\[
a_n = 2a_{n-1} + c
\]

\[
= 2c(2^{n-1} - 1) + c \text{ by I.H.}
\]

\[
= c2^n - 2c + c = c(2^n - 1)
\]

Towers of Hanoi

• Figures
  – Figs 1-4
    • problem size n-1
  – Figs 4-5
    • Constant work
  – Figs 5-7
    • problem size n-1

Example 3

• \( a_n = 2a_{n/2} + c \)

• \( a_1 = c \)

• Solve recurrence
• Then prove true by induction
• What is this an example of?
Example 3

\[a_1 = c\]

\[a_n = 2 \cdot a_{n/2} + c\]

= 

= 

= 

... 

= 

Let \( n = 2^k \)

Example 3

\[a_1 = c\]

\[a_n = 2 \cdot a_{n/2} + c\]

= 

= 

= 

... 

= \[2^k \cdot a_{n/2^k} + c(2^k - 1)\]

Example 3

= \[n \cdot a_1 + c(n - 1)\]

= \[n \cdot (c) + cn - c\]

= \[2cn - c\]

Example 3

\[a_n = 2cn - c\]
Prove by induction

Basis: \( a_1 = c \)
\[ a_n = 2cn - c \]
\( a_1 = \)

 Assume: \( a_k = 2c^k - c \) for \( k < n \)
\[ a_n = 2 \times a_{n/2} + c \]
= 
= 

Traversal in binary search tree
preorder, postorder, inorder

Example 4

- \( a_n = 2a_{n/2} + cn \)
- \( a_1 = c \)

- Solve recurrence
- Then prove true by induction
- What is this an example of?
Example 4

\[ a_1 = c \]
\[ a_n = 2a_{n/2} + cn \]
\[ = \]
\[ = \]
\[ = \]
\[ = \]
\[ \ldots \]
\[ = \]

Let \( n/2^k = 1 \) → \( k = \log_2 n \)

Example 4

\[ a_1 = c \]
\[ a_n = 2a_{n/2} + cn \]
\[ = 2[2a_{n/4} + cn/2] + cn \]
\[ = 2^2 a_{n/4} + 2cn \]
\[ = 2^2 [2a_{n/8} + cn/4] + 2cn \]
\[ = 2^3 a_{n/8} + 3cn \]
\[ \ldots \]
\[ = 2^k a_{n/2^k} + kcn \]

Example 4

- \( = n a_1 + (\log_2 n)cn \)
- \( = nc + cn\log_2 n \)
- \( = cn(1 + \log_2 n) \)
Proof by induction

Basis: $a_1 = c$

$a_1 = c$

Assume: $a_k = c(1 + \log_2 k)$ for all $k < n$

$a_n = 2 * a_{n/2} + cn$

$= 2 * [cn/2 + cn/2 \log_2 n/2] + cn$

$= cn + cn (\log_2 n + \log_2 \frac{1}{2}) + cn$

$= cn(2 + \log_2 n - 1) = cn(1 + \log_2 n)$

Proof by induction

Basis: $a_1 = c$

$a_1 = cn(1 + \log_2 n)$

$= c(1)(1 + \log_2 1)$ for $n=1$

$= c * (1 + 0) = c$ ✔

Assume: $a_k = ck + ck \log_2 k$ for all $k < n$

$a_n = 2 * a_{n/2} + cn$

$= 2 * [cn/2 + cn/2 \log_2 n/2] + cn$

$= cn + cn (\log_2 n + \log_2 \frac{1}{2}) + cn$

$= cn(2 + \log_2 n - 1) = cn(1 + \log_2 n)$ ✔

MergeSort

- $n \log n$

Definition

- A linear homogeneous recurrence relation of degree $k$ with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

- Where $c_i$ are real numbers and $c_k \neq 0$
8.2 - Theorem 1

Let $c_1$ and $c_2$ be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots $r_1$ and $r_2$. Then the sequence $\{a_n\}$ is a solution of the recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

for all $n$ where $\alpha_1$ and $\alpha_2$ are constants.

Example

- What is the solution to the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2, a_1 = 7$?

$$a_n = \frac{a_{n-1} + 2a_{n-2}}{r^2 - c_1 r - c_2 = 0}$$

Note $c_1 = \_ , c_2 = _-$

$$\rightarrow r_1 = , r_2 =$$

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

$$a_n = 2 = a_0 = \quad sub \ n = 0$$

$$7 = a_1 = \quad sub \ n = 1$$

$$\rightarrow \quad \alpha_1 = , \quad \alpha_2 =$$

$$a_n =$$

8.2 - Theorem 2

- Let $c_1$ and $c_2$ be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root $r_0$. A sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if

$$a_n =\alpha_1 r_0^n + \alpha_2 n r_0^n$$

for all $n$ where $\alpha_1$ and $\alpha_2$ are constants.

Example

- What is the solution to the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2, a_1 = 7$?

$$a_n = \frac{a_{n-1} + 2a_{n-2}}{r^2 - c_1 r - c_2 = 0}$$

Note $c_1 = \_ , c_2 = _-$

$$\rightarrow r_1 = , r_2 =$$

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

$$a_n = 2 = a_0 = \quad sub \ n = 0$$

$$7 = a_1 = \quad sub \ n = 1$$

$$\rightarrow \quad \alpha_1 = , \quad \alpha_2 =$$

$$a_n =$$
Many other theorems

- See theorems 2-6 in Chapter 8.2

Theorem 6

Suppose that \( \{a_n\} \) satisfies the linear nonhomogeneous recurrence relation

\[
a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)
\]

where \( c_1, c_2, \ldots, c_k \) are real numbers, and

\[
F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,
\]

where \( b_0, b_1, \ldots, b_t \) and \( s \) are real numbers. When \( s \) is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

\[
(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n
\]

Theorem 6 (cont)

When \( s \) is a root of this characteristic equation and its multiplicity is \( m \), there is a particular solution of the form

\[
n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n
\]

Theorem 1 in 8.3

Theorem 1: Let \( f \) be an increasing function that satisfies the recurrence relation

\[
f(n) = af(n/b) + c
\]

whenever \( n \) is divisible by \( b \), where \( a \geq 1 \), \( b \) is an integer greater than 1, and \( c \) is a positive real number. Then

\[f(n) \text{ is } \begin{cases} O(n^{\log_a b}) & \text{if } a > 1 \\ O(\log n) & \text{if } a = 1. \end{cases}\]

Furthermore, when \( n = b^k \) and \( a \neq 1 \), where \( k \) is a positive integer,

\[
f(n) = C_1 n^{\log_a b} + C_2
\]

where \( C_1 = f(1) + c/(a-1) \) and \( C_2 = -c/(a-1) \).
Master Theorem in 8.3

**Theorem 2. Master Theorem:** Let $f$ be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + cn^d$$

whenever $n = b^k$, where $k$ is a positive integer greater than 1, and $c$ and $d$ are real numbers with $c$ positive and $d$ nonnegative. Then

$$f(n) \begin{cases} O(n^d) & \text{if } a < b^d, \\ O(n^d \log n) & \text{if } a = b^d, \\ O(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$