Discrete Probability IV

COMPSCI 230 — Discrete Math

April 14, 2016
Outline

1. Random Variables and Probability Distributions
2. Expectation
3. Repeated Bernoulli Trials
From Outcomes to Numbers

- We often associate numbers to outcomes.
- Example: Coin flip, bet on $H$
  - Win $W = g = \$10$ if head, win $W = \ell = -\$10$ if tail.
- How much do I win on average if $p = P(H) = 0.6$?
- Let the answer be $a$.
- Approximate frequentist interpretation: If I play the game 1000 times, then I win close to $1000a$.
- However, we play once.
- We need to tie $g$ to $H$ and $\ell$ to $T$. 
Random Variables

- We need to tie $g$ to $H$ and $\ell$ to $T$
- $H, T$ are in the sample space $\mathbb{S}$
- $g, \ell$ are in $\mathbb{R}$
- Any function $W = f(O) : \mathbb{S} \rightarrow \mathbb{R}$ is called a **random variable**
- *It is neither random, nor a variable*
- It is a deterministic function of the outcome
- If you just look at the *image* ($W = g$ or $W = \ell$) as a variable then its value varies randomly
- Hence the (arguably confusing) name
- In the example,
  \[
  W : \{H, T\} \rightarrow \mathbb{R}
  \]
  is defined as
  \[
  W(H) = g \quad \text{and} \quad W(T) = \ell
  \]
Probability Distribution

- A probability function defined on a random variable is called a **probability distribution**:
  \[ P(X = x) = \sum_{O \in \mathbb{S} : X(O) = x} P(O) \]

- \( P(X = x_a) = P(O_1) \), \( P(X = x_b) = P(O_2) + P(O_3) \)

- Example 1: Roll of one fair die, \( x = X(O) = (O \mod 2) \)
  - \( X(2) = X(4) = X(6) = 0 \)
  - \( X(1) = X(3) = X(5) = 1 \)
  - \( P(X = 0) = P(X = 1) = 1/2 \)

- Example 2: Money won when betting on \( H \) in a coin flip,
  \[ W(O) = \begin{cases} 
  g & \text{for } O = H \\
  \ell & \text{for } O = T 
  \end{cases} \]
  - \( P(W = g) = 0.6 \)
  - \( P(W = \ell) = 0.4 \)
Expectation

- I win $W = g = $10 with probability $P(H) = p = 0.6$
- I win $W = \ell = -$10 with probability $P(T) = q = (1 - p) = 0.4$
- My **expected win** is defined as
  
  $$a = \mathbb{E}[W] = \mathbb{E}[f(O)] = P(H)f(H) + P(T)f(T) = pf(H) + qf(T) = pg + (1 - p)\ell = 0.6 \cdot 10 + 0.4 \cdot (-10) = $2$$

- If I play the game 1000 times, then I win close to
  
  $$1000\mathbb{E}[W] = $2000$$

- However, we play once: Expected win is $2$
- More generally, the **expected value** (or **expectation**) of random variable $W = f(O)$ is

  $$E[W] = E[f(O)] = \sum_{O \in \mathcal{S}} P(O)f(O) = \sum_{x \in \mathbb{R}} x \sum_{O \in \mathcal{S} : f(O) = x} P(O)$$

- Also applies to compound experiments ($O$ is a tuple)
Expectation is Linear

- \( \mathbb{E}[a U + b V] = a \mathbb{E}[U] + b \mathbb{E}[V] \) because

\[
\mathbb{E}[a U + b V] = \sum_{O \in \mathcal{S}} P(O) \left[ a U(O) + b V(O) \right]
\]

\[
= a \sum_{O \in \mathcal{S}} P(O) U(O) + b \sum_{O \in \mathcal{S}} P(O) V(O)
\]

\[
= a \mathbb{E}[U] + b \mathbb{E}[V]
\]

- Example: Upton wins or loses 8 pounds sterling. Valerie wins or loses 10 dollars
  - \( \mathbb{E}[U] = 0.6 \cdot 8 - 0.4 \cdot 8 = 1.6 \) pounds
  - \( \mathbb{E}[V] = 0.6 \cdot 10 - 0.4 \cdot 10 = 2 \) dollars
  - Exchange rate: 1.43 dollars per pound sterling
  - Combined win: \( 1.43 \mathbb{E}[U] + \mathbb{E}[V] = 1.43 \cdot 1.6 + 2 = \$4.288 \)
  - If the exchange rate changes, we do not need to recompute \( \mathbb{E}[U], \mathbb{E}[V] \)
Repeated Bernoulli Trials

Sequences of Bernoulli Trials

- A repeated coin flip is an example of a Bernoulli trial:
  - Two outcomes per repetition
  - Fixed probability $p$ of “success”
  - Repetition outcomes are independent
- “Trial” often used for “repetition”
- This is an unbounded repeated experiment
- Outcome $C = (C_1, C_2, ...) \ (\text{Python generator?})$
- Sample space $\mathcal{S}^\infty = \mathcal{S} \times \mathcal{S} \times ... \ \text{where} \ \mathcal{S} = \{H, T\}$
- The infinite sequence $C$ is one outcome of the repeated experiment
- So we can define random variables

\[
X : \mathcal{S}^\infty \to \mathbb{R}
\]

- Example:

\[
N(C) = n \ \text{iff the first} \ H \ \text{is in trial} \ n \in \mathbb{N}
\]
Expected Number of Trials to First Success

- $N(C) = n$ iff the first $H$ is in trial $n \in \mathbb{N}$
- $N = n$ iff $n - 1$ $T$s are followed by one $H$
- [and what happens after that does not matter]
- Example: $n = 4$. $(T, T, T, H, ...)$
- Because of independence, $P(N = n) = q^{n-1}p$
- $\mathbb{E}[N]$: expected number of trials until the first success
- How many times do you need to flip a coin on average until it comes up $H$?
A Summation Lemma

• For $0 < x < 1$

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1 - x)^2}$$

• Proof: We know that for $x \neq 1$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}$$

$$\sum_{n=0}^{\infty} nx^n = x \sum_{n=0}^{\infty} nx^{n-1} = x \sum_{n=0}^{\infty} \frac{d}{dx} x^n = x \frac{d}{dx} \sum_{n=0}^{\infty} x^n$$

$$= x \frac{d}{dx} \frac{1}{1 - x} = x \frac{1}{(1 - x)^2} = \frac{x}{(1 - x)^2}$$
Expected Number of Trials to First Success

- The expected number of trials to first success in a Bernoulli experiment with success probability $p$ is

$$
\mathbb{E}[N] = \frac{1}{p}
$$

- Proof: Let $N(C) = n$ iff the first $H$ is in trial $n \in \mathbb{N}$

$$
\mathbb{E}[N] = \sum_{C \in \mathcal{S}^\infty} N(C) P(C) = \sum_{n=1}^\infty n P(N = n)
$$

$$
= \sum_{n=1}^\infty n q^{n-1} p = \sum_{n=0}^\infty n q^{n-1} p = \frac{p}{q} \sum_{n=0}^\infty n q^n
$$

(from lemma) $$
= \frac{p}{q} \frac{q}{(1 - q)^2} = \frac{p}{q} \frac{q}{p^2} = \frac{1}{p} \quad \square
$$

- Application case study: analysis of hashing