- Two player games
  - e.g. rock paper scissors
  - can represent outcome of RPS as a matrix

<table>
<thead>
<tr>
<th></th>
<th>R</th>
<th>P</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>P</td>
<td>1</td>
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</tr>
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<td>S</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Alice entry in matrix: result of this game for Alice

Goal of players: Alice wants to maximize the winning probability
Bob wants to minimize Alice's winning probability

Zero-sum games: 1 player wins = the other player loses

- Strategy: probability distribution over actions.

- "value" of the game: $E[\text{payoff}]$

$$
= \sum_{i \in \text{RPS}} \sum_{j \in \text{RPS}} P_i \cdot P_j \cdot A_{ij}
$$

- example: if $P_R = 1$, $Q_S = 1$, Payoff = 1 (Alice always wins)
  - if $P_R = 1$, $Q_R = Q_P = \frac{1}{2}$, $Q_S = \frac{1}{2}$, Payoff = $\frac{1}{2}$
    (Alice is expected to get 1 net win in every 4 games)

A B C UNC
\[
\begin{array}{ccc}
A & 3 & 1 & -1 \\
B & -2 & 3 & 2 \\
C & 1 & -2 & 4 \\
\end{array}
\]

Duke

Finding a strategy for Duke: play A w.p. \(x_1\), play B w.p. \(x_2\), play C w.p. \(x_3\).

Want: no matter what UNC does, always get an expected payoff of at least \(x_4\).

- Using linear program:
  - Constraints
    1. Probability constraints
      \[x_1, x_2, x_3 \geq 0, \quad x_1 + x_2 + x_3 = 1\]
    2. Payoff constraints
      \[3x_1 - 2x_2 + x_3 \geq x_4 \quad \text{UNC plays A} \]
      \[x_1 + 3x_2 - 2x_3 \geq x_4 \quad \text{B} \]
      \[-x_1 + 2x_2 + 4x_3 \geq x_4 \quad \text{C} \]
  - Objective function
    \[
    \max \ x_4
    \]
  - Optimal solution \(\iff\) best strategy
  - UNC's strategy
    - Play A w.p. \(y_1\), B w.p. \(y_2\), C w.p. \(y_3\)
    - Expected payoff (for Duke) \(y_4\) (Want to minimize \(y_4\))
      \[y_1, y_2, y_3 \geq 0, \quad y_1 + y_2 + y_3 = 1\]
      \[3y_1 + y_2 - y_3 \leq y_4 \quad \text{(Duke plays A)} \]
      \[-2y_1 + 3y_2 + 2y_3 \leq y_4 \quad \text{B} \]
      \[y_1 - 2y_2 + 4y_3 \leq y_4 \quad \text{C} \]
      \[
      \min y_4
      \]
  - Q: \(y_4\) be optimal value for (1), \(y_4\) be optimal value for (2)
Which is larger?

- A: $X_4 = Y_4$

- Why? If $X_4 > Y_4$ we know if both of them play optimal strategy
  expected payoff $\geq X_4$ (guarantee of first LP)
  expected payoff $\leq Y_4$ (guarantee of second LP)

(Weak duality)

if $X_4 < Y_4$ contradiction is more complicated. But it's also not possible.

$X_4 = Y_4$ "Strong" duality for two-player games.

- Solution to LPs: $X_1 = \frac{9}{19}, X_2 = \frac{6}{19}, X_3 = \frac{4}{19}, X_4 = 1$ indeed the same.

  $Y_1 = Y_2 = Y_3 = \frac{1}{3}, Y_4 = 1$

- Theorem (minimax theorem): For any two-player zero-sum game, there is always a pair of optimal strategies and a value $V$. If player A plays optimal strategy, can always guarantee payoff $\geq V$.
  If B plays optimal strategy, can always guarantee payoff $\leq V$.

(RPS, $V = 0$ Duke-UVAE value 1)

- Duality for Linear Program
  Recall: canonical form

  $$\min \ c^T x$$
  s.t. $A x \geq b$
  $x \geq 0$

  e.g. $\min 2X_1 - 3X_2 + X_3$

  $X_1 - X_2 \geq 1$ (1)
  $X_2 - 2X_3 \geq 2$ (2)
  $-X_1 - X_2 - X_3 \geq -7$ (3)
  $X_1, X_2, X_3 \geq 0$

  Q: How can I convince you that optimal solution has value at least -1.

  [Verify optimal $\leq -1$: can be done by a feasible solution]

  $(X_1 = 4, X_2 = 3, X_3 = 0)$

  [because this is feasible, optimal solution $\leq -1$]

  Idea for proving optimal solution $> -1$
- Because this is feasible, optimal solution \( \leq -1 \)

**Idea for proving optimal solution \( \geq -1 \)**

\[
2.5X(1) + 0.5X(3) \\
\Rightarrow 2X_1 - 3X_2 - 0.5X_3 \geq -1 \\
2X_1 - 3X_2 + X_3 \geq 2X_1 - 3X_2 - 0.5X_3 \geq -1 \\
X_3 \geq 0
\]

Optimal solution is at least \(-1\)!

- General way for finding these proofs?

- Idea: can actually find the proof using a LP!

**Primal LP**

For each constraint, have a variable \( y_i \);

\[
\begin{align*}
\text{min} & \quad c^T x \\
\text{s.t.} & \quad \sum_{i=1}^{m} y_i \mathbf{a}_i \geq b_i \\
& \quad Y_1 x(1) + Y_2 x(2) + \ldots + Y_m x(m) \\
& \quad \sum_{i=1}^{m} y_i = 0 \\
& \quad x_i \geq 0
\end{align*}
\]

Want to show \( c^T x \geq (\sum_{i=1}^{m} y_i \mathbf{a}_i)^T x \geq (\sum_{i=1}^{m} y_i b_i) \)

To show this need \( c_j \geq \sum_{i=1}^{m} y_i \mathbf{a}_i(j) \)

In order to find best proof of this kind

\[
\begin{align*}
Y_i & \geq 0 \\
C_j & \geq \sum_{i=1}^{m} y_i \mathbf{a}_i(j) \\
\max & \quad \sum_{i=1}^{m} y_i b_i
\end{align*}
\]

- More succinctly

\[
\begin{align*}
\min & \quad c^T x \\
A x & \geq b \\
& \quad x \geq 0
\end{align*} \quad \iff \quad \begin{align*}
\max & \quad b^T y \\
A^T y & \leq c \\
& \quad y \geq 0
\end{align*}
\]

**Primal**

**Dual**