This homework assignment will not be collected and will not be graded. A solution sample is available on the class homework page. In your own interest, complete the assignment by the due date. Also, do not peek at the solutions until you have made an honest effort to answer all the questions below.

Exam Details

The second midterm exam covers all topics, textbook pages, and handouts listed under GEOMETRY and IMAGE FORMATION on the class syllabus web page, plus sections 6.5.2 and 5.1.1 of the textbook, and anything covered in homework assignments 4, 5, and 6. While midterms are not meant to be explicitly cumulative, you may need knowledge from material covered in midterm 1 to answer some of the questions in midterm 2.

The exam will be closed book, closed notes, and no calculator or any other electronic device is allowed. The number of questions on the exam will be similar to that in this assignment, and the time allotted for the exam will be one hour.

At the exam, you will be asked to keep only the exam text, one pencil or pen, and one eraser within reach. Please comply as soon as you get to your seat, in order to save time. There will be space for your answer under each question.

How to Answer

The questions below give you an idea of the flavor of questions that may come up in the second midterm exam, and of the expected format of your answers.

When you try answering the questions below, you may want to write down some detail for your own clarity. On the exam, on the other hand, just answer the questions as briefly as possible, and give explanations only when they are requested, or when you are unsure of your answer.

Sample Questions

1. Write a formula for the rotation vector $r$ corresponding to a clockwise 45-degree rotation around a line through the origin and parallel to the vector $v = [1, 1, 1]^T$?

2. Compute the numerical values in the vector $c$ that is the cross product of the vectors $a = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

3. Write the $3 \times 3$ matrix $A$ such that $Ax$ is equal for all $x \in \mathbb{R}^3$ to the cross product $a \times x$ where $a = [1, 2, 3]^T$.

4. Let $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ be a nonzero vector. The product $Ux$ where $U = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$

is the same as $u \times x$.

What is the rank of $U$? Explain your answer in geometric terms.

5. A rigid transformation in $\mathbb{R}^3$ has the following expression in Euclidean coordinates:

$y = Rx + t$  where  $RR^T = R^T R = I_3$

and $t$ is a vector in $\mathbb{R}^3$. Prove that rigid transformations in $\mathbb{R}^3$ preserve Euclidean distance.
6. The figure below shows a world Cartesian reference frame in two dimensions with origin $O$ and axes $(\xi_1, \xi_2)$ and a camera Cartesian reference frame with origin $C$ and axes identified by the orthogonal unit vectors $i$ and $j$, whose components are specified in world coordinates.

[Diagram of two coordinate systems]

Write the values in a matrix $R$ and a vector $t$ such that if $\xi$ is the vector of world coordinates for a point $P$ on the plane, then the vector of camera coordinates for $P$ is

$$x = R\xi + t.$$  

Formulas such as $\sqrt{2}/5$ in your answer are fine, you need not give decimal values.

7. What is the vector $v$ of the Euclidean coordinates of the point with the following homogeneous coordinates?

$$h = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

8. A certain rigid transformation in $\mathbb{R}^3$ is performed through the following two steps:

$$x' = R_1 x + t_1 \quad \text{followed by} \quad x'' = R_2 x' + t_2.$$  

Write expressions for the entries of a single matrix $g$ that describes this rigid transformation in homogeneous coordinates.

9. Write the thin-lens equation and define the variables that appear in it as precisely as you can.

10. An object in front of a thin lens with focal length $f$ is in focus when the focal distance is $z$. Write a formula for the distance of the object from the center of the lens, measured along the optical axis.

11. Why is the image plane placed in front of the center of projection in the mathematical model of a pinhole camera?

12. An ideal pinhole camera images a point $P$ in the world at image coordinates $(x, y)$ in units of focal distance relative to the principal point. The distance between the point and the camera pinhole $O$ is measured with a tape measure stretched between $O$ and $P$, and is found to be $d$ meters. Write formulas for the $X, Y, Z$ coordinates of $P$ in terms of $x, y$ and $d$ and in the standard reference frame centered at the pinhole. Include units of measure in your result.

13. What is the focal length needed to achieve a horizontal field of view of $\phi$ radians with a camera sensor that is $w$ millimeters wide?

14. An ideal perspective camera free of distortion has square pixels with 100 pixels per linear millimeter of sensor, no skew, principal point $o = (470, 350)$ pixels, and focal length $f = 10$ mm. Write the numerical values in the homogeneous-coordinate projection matrix $\Pi$ for this camera in its own reference frame (that is, the exterior calibration matrix $g$ is the identity). What is the unit of measure for the entries of $\Pi$?
15. Are undistorted coordinates \((x, y)\) even or odd functions of the distorted coordinates \((x_d, y_d)\) when the origin of the reference frame is the principal point of the image? Why?

16. Define the essential matrix \(E\) in terms of camera rotation \(R\) and translation \(t\). You may use the following notation

\[
[t]_x = \begin{bmatrix}
0 & -t_3 & t_2 \\
t_3 & 0 & -t_1 \\
-t_2 & t_1 & 0
\end{bmatrix}
\]

where \(t = \begin{bmatrix}
t_1 \\
t_2 \\
t_3
\end{bmatrix}\).

17. Define the epipole \(e_1\) in image 1 for a pair of cameras numbered 1 and 2.