Tutorial 5: Discrete Probability I

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Feb 15, 2008

Reference: http://www.cs.duke.edu/courses/fall07/cps102/lecture11.ppt
Language of Probability

The formal language of probability is a very important tool in describing and analyzing probability distribution.
Probability Space

- A Probability space has 3 elements:
  - Sample Space : $\Omega$
    - All the possible individual outcomes of an experiment
  - Event Space : $\mathcal{F}$
    - Set of all possible subsets of elements taken from $\Omega$
  - Probability Measure : $P$
    - A mapping from event space to real numbers such that for any $E$ and $F$ from $\mathcal{F}$
      - $P(E) \geq 0$
      - $P(\Omega) = 1$
      - $P(E \text{ union } F) = P(E) + P(F)$

- We can write a probability space as $(\Omega, \mathcal{F}, P)$
Sample Space: $\Omega$

$p(x) = 0.2$

probability of $x$

Sample space
Any set $E \subseteq \Omega$ is called an event

$$Pr_{D}[E] = \sum_{x \in E} p(x)$$

$$Pr_{D}[E] = 0.4$$
Uniform Distribution

If each element has equal probability, the distribution is said to be uniform

\[ \Pr_D[E] = \sum_{x \in E} p(x) = \frac{|E|}{|\Omega|} \]
A fair coin is tossed 100 times in a row.

What is the probability that we get exactly half heads?
The sample space $\Omega$ is the set of all outcomes $\{H,T\}^{100}$. Each sequence in $\Omega$ is equally likely, and hence has probability $1/|\Omega|=1/2^{100}$. 

Using the Language
$\Omega = \text{all sequences of 100 tosses}$

$x = \text{HHTTTT…….TH}$

$p(x) = \frac{1}{|\Omega|}$
Set of all $2^{100}$ sequences $\{H,T\}^{100}$

Event $E = \text{Set of sequences with 50 } H\text{'s and 50 } T\text{'s}$

Probability of event $E = \text{proportion of } E \text{ in } \Omega$

$$\binom{100}{50} / 2^{100}$$
Suppose we roll a white die and a black die.

What is the probability that sum is 7 or 11?
Same Methodology!

\[ \Omega = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\} \]

\[ \text{Pr}[E] = \frac{|E|}{|\Omega|} = \text{proportion of } E \text{ in } S = \frac{8}{36} \]
23 people are in a room

Suppose that all possible birthdays are equally likely

What is the probability that two people will have the same birthday?
And The Same Methods Again!

Sample space $\Omega = \{1, 2, 3, \ldots, 366\}^{23}$

Event $E = \{ x \in W \mid \text{two numbers in } x \text{ are same} \}$

What is $|E|$? Count $|\overline{E}|$ instead!
$E = \text{all sequences in } S \text{ that have no repeated numbers}$

$|E| = (366)(365)\ldots(344)$

$|\Omega| = 366^{23}$

$\frac{|E|}{|\Omega|} = 0.494\ldots$

$\frac{|E|}{|\Omega|} = 0.506\ldots$
More Language Of Probability

The probability of event $A$ given event $B$ is written $\Pr[ A | B ]$ and is defined to be:

$$
\frac{\Pr[ A \cap B ]}{\Pr[ B ]}
$$

proportion of $A \cap B$ to $B$
Suppose we roll a white die and black die

What is the probability that the white is 1 given that the total is 7?

- event A = \{white die = 1\}
- event B = \{total = 7\}
\[ \Omega = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \} \]

\[
\text{event } A = \{ \text{white die } = 1 \} \\
\text{event } B = \{ \text{total } = 7 \}
\]

\[
\Pr[A | B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{|A \cap B|}{|B|} = \frac{1/36}{1/6}
\]
Independence!

A and B are independent events if

\[ \Pr[ A | B ] = \Pr[ A ] \]

\[ \iff \]

\[ \Pr[ A \cap B ] = \Pr[ A ] \Pr[ B ] \]

\[ \iff \]

\[ \Pr[ B | A ] = \Pr[ B ] \]
Independence!

$A_1, A_2, \ldots, A_k$ are **independent events** if knowing if some of them occurred does not change the probability of any of the others occurring.

- E.g., $\{A_1, A_2, A_3\}$ are independent events if:
  - $\Pr[A_1 | A_2 \cap A_3] = \Pr[A_1]$  
  - $\Pr[A_2 | A_1 \cap A_3] = \Pr[A_2]$  
  - $\Pr[A_3 | A_1 \cap A_2] = \Pr[A_3]$

\[
\begin{align*}
\Pr[A_1 | A_2 ] &= \Pr[A_1] \\
\Pr[A_2 | A_1 ] &= \Pr[A_2] \\
\Pr[A_3 | A_1 ] &= \Pr[A_3] \\
\Pr[A_1 | A_3 ] &= \Pr[A_1] \\
\Pr[A_2 | A_3 ] &= \Pr[A_2] \\
\Pr[A_3 | A_2 ] &= \Pr[A_3]
\end{align*}
\]
Silver and Gold

One bag has two silver coins, another has two gold coins, and the third has one of each.

One bag is selected at random. One coin from it is selected at random. It turns out to be gold.

What is the probability that the other coin is gold?
Let $G_1$ be the event that the first coin is gold
$\Pr[G_1] = 1/2$

Let $G_2$ be the event that the second coin is gold
$\Pr[G_2 | G_1] = \Pr[G_1 \text{ and } G_2] / \Pr[G_1]$

$\quad = (1/3) / (1/2)$

$\quad = 2/3$

Note: $G_1$ and $G_2$ are not independent
The Monty Hall Problem

- http://www.youtube.com/watch?v=mhlc7peGIgG
The Monty Hall Problem

1. Player picks car
   - Host reveals Goat A or Goat B
   - Switching loses.

2. Player picks Goat A
   - Host must reveal Goat B
   - Switching wins.

3. Player picks Goat B
   - Host must reveal Goat A
   - Switching wins.

http://en.wikipedia.org/wiki/Monty_Hall_problem
The Monty Hall Problem

Contestant chooses:

Host reveals:

<table>
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<tr>
<th>Contestant chooses</th>
<th>Goats Revealed</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goat A</td>
<td>Goat B</td>
<td>1/3</td>
</tr>
<tr>
<td>Goat B</td>
<td>Goat A</td>
<td>1/3</td>
</tr>
<tr>
<td>Car</td>
<td>Goat B, Goat A</td>
<td>1/3</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>Stay:</th>
<th>Goats Revealed</th>
<th>Probability</th>
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</thead>
<tbody>
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<td>Goat A</td>
<td>1/6</td>
</tr>
<tr>
<td>Goat B</td>
<td>Goat B</td>
<td>1/6</td>
</tr>
<tr>
<td>Car</td>
<td>Car</td>
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<th>Switch:</th>
<th>Goats Revealed</th>
<th>Probability</th>
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<tbody>
<tr>
<td>Car</td>
<td>Goat A</td>
<td>1/6</td>
</tr>
<tr>
<td>Car</td>
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The Monty Hall Problem

- Still doubt?
- Try playing the game online:
Thank you!

Q&A