What is game theory?

- Study of settings where multiple parties (agents) each have
  - different preferences (utility functions),
  - different actions

- Each agent’s utility (potentially) depends on all agents’ actions
  - What is optimal for one agent depends on what other agents do
  - Can be circular

- Game theory studies how agents can rationally form beliefs over what other agents will do, and (hence) how agents should act

- Can be useful for acting as well as (potentially) predicting behavior of others

- Not necessarily descriptive
Real World Game Theory Examples

- War
- Auctions
- Animal behavior
- Networking protocols, peer to peer networking behavior
- Road traffic

- Mechanism design: Suppose we want people to do X? How do we engineer the situation so that they will act that way?

Rock-paper-scissors

<table>
<thead>
<tr>
<th></th>
<th>Rock</th>
<th>Paper</th>
<th>Scissors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rock</td>
<td>0,0</td>
<td>-1,1</td>
<td>1,-1</td>
</tr>
<tr>
<td>Paper</td>
<td>-1,1</td>
<td>0,0</td>
<td>-1,1</td>
</tr>
<tr>
<td>Scissors</td>
<td>1,-1</td>
<td>1,-1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Column player AKA player 2 (simultaneously) chooses a column
Row player AKA player 1 chooses a row
A row or column is called an action or (pure) strategy
Row player’s utility is always listed first, column player’s second

Zero-sum game: the utilities in each entry sum to 0 (or a constant)
Three-player game would be a 3D table with 3 utilities per entry, etc.
“Chicken”

- Two players drive cars towards each other
- If one player goes straight, that player wins
- If both go straight, they both die

<table>
<thead>
<tr>
<th></th>
<th>D</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>0,0</td>
<td>-1,1</td>
</tr>
<tr>
<td>S</td>
<td>1,-1</td>
<td>-5,-5</td>
</tr>
</tbody>
</table>

Rock-paper-scissors – Seinfeld variant

MICKEY: All right, rock beats paper!
(Mickey smacks Kramer’s hand for losing)
KRAMER: I thought paper covered rock.
MICKEY: Nah, rock flies right through paper.
KRAMER: What beats rock?
MICKEY: (looks at hand) Nothing beats rock.
Dominance

- Player $i$’s strategy $s_i$ strictly dominates $s_i'$ if
  - for any $s_{-i}$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$

- $s_i$ weakly dominates $s_i'$ if
  - for any $s_{-i}$, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$; and
  - for some $s_{-i}$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$

\[
\begin{array}{ccc}
0, 0 & 1, -1 & 1, -1 \\
-1, 1 & 0, 0 & -1, 1 \\
-1, 1 & 1, -1 & 0, 0 \\
\end{array}
\]

Prisoner’s Dilemma

- Pair of criminals has been caught
- District attorney has evidence to convict them of a minor crime (1 year in jail); knows that they committed a major crime together (3 years in jail) but cannot prove it
- Offers them a deal:
  - If both confess to the major crime, they each get a 1 year reduction
  - If only one confesses, that one gets 3 years reduction

\[
\begin{array}{cc}
\text{confess} & \text{don’t confess} \\
-2, -2 & 0, -3 \\
-3, 0 & -1, -1 \\
\end{array}
\]
“Should I buy an SUV?”

<table>
<thead>
<tr>
<th>purchasing + gas cost</th>
<th>accident cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>cost: 5</td>
<td>cost: 5</td>
</tr>
<tr>
<td>cost: 3</td>
<td>cost: 8</td>
</tr>
<tr>
<td>cost: 5</td>
<td>cost: 2</td>
</tr>
<tr>
<td>cost: 5</td>
<td>cost: 5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>-10, -10</th>
<th>-7, -11</th>
</tr>
</thead>
<tbody>
<tr>
<td>-11, -7</td>
<td>-8, -8</td>
</tr>
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</table>

“2/3 of the average” game

- Everyone writes down a number between 0 and 100
- Person closest to 2/3 of the average wins
- Example:
  - A says 50
  - B says 10
  - C says 90
  - Average(50, 10, 90) = 50
  - 2/3 of average = 33.33
  - A is closest (|50-33.33| = 16.67), so A wins
Iterated dominance

- Iterated dominance: remove (strictly/weakly) dominated strategy, repeat
- Iterated strict dominance on Seinfeld’s RPS:

\[
\begin{array}{ccc}
0, 0 & 1, -1 & 1, -1 \\
-1, 1 & 0, 0 & -1, 1 \\
-1, 1 & 1, -1 & 0, 0 \\
\end{array}
\]

Mixed strategies

- Mixed strategy for player i = probability distribution over player i’s (pure) strategies
- E.g. 1/3, 1/3, 1/3
- Example of dominance by a mixed strategy:

\[
\begin{array}{ccc}
3, 0 & 0, 0 \\
0, 0 & 3, 0 \\
1, 0 & 1, 0 \\
\end{array}
\]
Nash equilibrium [Nash 50]

- A vector of strategies (one for each player) = a strategy profile
- Strategy profile \((\sigma_1, \sigma_2, \ldots, \sigma_n)\) is a Nash equilibrium if each \(\sigma_i\) is a best response to \(\sigma_i\)
  - That is, for any \(i\), for any \(\sigma'_i\), \(u_i(\sigma_i, \sigma_i) \geq u_i(\sigma'_i, \sigma_i)\)
- Does not say anything about multiple agents changing their strategies at the same time
- In any (finite) game, at least one Nash equilibrium (possibly using mixed strategies) exists [Nash 50]

- (Note - singular: equilibrium, plural: equilibria)

Nash equilibria of “chicken”

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<td>-1, 1</td>
</tr>
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<td>S</td>
<td>1, -1</td>
<td>-5, -5</td>
</tr>
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</table>

- \((D, S)\) and \((S, D)\) are Nash equilibria
  - They are pure-strategy Nash equilibria: nobody randomizes
  - They are also strict Nash equilibria: changing your strategy will make you strictly worse off
- No other pure-strategy Nash equilibria
Rock-paper-scissors

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- Any pure-strategy Nash equilibria?
- But it has a mixed-strategy Nash equilibrium:
  - Both players put probability 1/3 on each action
- If the other player does this, every action will give you expected utility 0 - Might as well randomize

Nash equilibria of “chicken”...

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- Is there a Nash equilibrium that uses mixed strategies -- say, where player 1 uses a mixed strategy?
- If a mixed strategy is a best response, then all of the pure strategies that it randomizes over must also be best responses
- So we need to make player 1 indifferent between D and S
- Player 1’s utility for playing D = -pD
  - pD = probability that column player plays S
- Player 1’s utility for playing S = pS - 5pD - 1 - 6pD
  - So we need -pD = 1 - 6pS which means pS = 1/6
  - Then, player 2 needs to be indifferent as well
- Mixed-strategy Nash equilibrium: ((4/5 D, 1/5 S), (4/5 D, 1/5 S))
  - People may die! Expected utility -1/6 for each player
Computational Issues

• Zero-sum games can be solved efficiently as linear programs (see slides from earlier in the semester)

• General sum games may require exponential time (in # of actions) to find a single equilibrium (no known efficient algorithm and good reasons to suspect that none exists)

• Some better news: Despite bad worst-case complexity, many games can be solved quickly

Game Theory Issues

• How descriptive is game theory?
  – Some evidence that people play equilibria
  – Also, some evidence that people act irrationally
  – If it is computationally intractable to solve for equilibria of large games, seems unlikely that people are doing this

• How reasonable is game theory?
  – Are payoffs known?
  – Are situations really simultaneous move with no information about how the other player will act?
  – Are situations really single-shot
Extensions

- Partial information
- Uncertainty about the game parameters, e.g., payoffs (Bayesian games)
- Multistep games with distributions over next states (game theory + MDPs = stochastic games)
- Multistep + partial information (Partially observable stochastic games)

- Game theory is so general, that it can encompass essentially all aspects of strategic, multiagent behavior, e.g., negotiating, threats, bluffs, coalitions, bribes, etc.