Splay Trees (Sleator - Tarjan)

- Binary search tree (left subtree <= node <= right subtree) [INSERT/DELETE/FIND]
- Height balancing not explicit but performed through `splaying` operations
- Comparison to other height balanced trees
  - $O(\log n)$ amortized (as against worst-case) running time per operation
  - Simpler `splay` operation
  - Has additional properties (see [PIET9])

SPLAY operations ($\text{splay}(v)$)

Case 1: $v$ is in the child of the root node

- Left child
- (right rotate)

- Right child
- (left rotate)

Case 2: $v$ is not a child of the root node

- LL (reversible with LR)
- (not reversible)

Analysis

Let $x$ be a node

- $s_x =$ # of descendants of $x$ including $x$ itself
- Range $r_x =$ $\log s_x$
- Potential $\Phi_i = \sum r_x$

Lemma: amortized cost of single level splay for node $x$

\[
\sum (r'(x) - r(x)) + 1
\]

where $r'$ and $r$ are ranks of $x$
Proof:

\[ R(x) = \begin{cases} \lambda \rightarrow \lambda & \text{if } y \neq x \\
2 \lambda \rightarrow \lambda & \text{if } y = x \end{cases} \]

**Lemma**: Amortized cost of an LL (or RR) rotation at \( x \)

\[ \leq 3 (r'(x) - r(x)) \]

Proof:

\[ \Delta \phi = r'(z) + r'(y) - r(x) - r(y) \]

\[ = (r'(z) - r(x)) + (r'(y) - r(y)) \leq r'(z) - r(x) \text{ since } r'(y) \leq r(y) \]

Using concavity of log function,

\[ r'(z) = \frac{\log s_x + \log s'_z}{2} \geq \frac{\log s_x}{2} \]

\[ \Rightarrow \ r'(z) \leq 2 r'(x) - 2 - r(x) \]

\[ \Delta \phi \leq 3 (r'(x) - r(x)) - 2 - r(x) - 2 r(x) = 3 (r'(x) - r(x)) - 2 \]

Overall, actual cost + \( \Delta \phi \) \[ \leq 3 (r'(x) - r(x)) \]

**Lemma**: Amortized cost of an LR (or RL) rotation at vertex \( x \)

\[ \leq 2 (r'(x) - r(x)) \]
\[
\Delta \phi = (r'(y) + r'(z)) - (r(x) + r(y) + r(z)) \\
\leq (r'(y) + r'(z)) - (r(x) + r(y)) \quad [r'(x) = r(z)] \\
\leq (r'(y) + r'(z)) - (r(x) + r(z)) \quad [r(y) \geq r(x)] \\
\text{by concavity of log function,} \\
\frac{r'(y) + r'(z)}{2} \leq \log \left( \frac{s'(y) + s'(z)}{2} \right) \leq \log \left( \frac{s'(x)}{2} \right) \\
= r'(x) - 1 \\
\Rightarrow r'(y) + r'(z) \leq 2r'(x) - 2 \\
\text{Thus, } \Delta \phi \leq 2r'(x) - 2 - 2r(x) \\
\text{Overall, actual cost} + \Delta \phi \leq 2(r'(x) - r(x))
\]

Operations:
- **Insert**: insert in BST and splay up to root
- **Delete**: find pred/succ and splay up to root
- **Find**: find element and splay up to root

Theorem: The amortized cost of each operation in a splay tree is \( O(\log n) \).

Proof:
\[
\text{amortized cost} \leq 3 \left( r^{t+1}(x) - r^t(x) \right) + 1 \\
= 3 \left( r^{\text{final}}(x) - r^{\text{init}}(x) \right) + 1 \\
= O \left( \frac{1}{\log n} \right)
\]