Lecture 3.4
Tuesday, September 10, 2013
12:01 AM

The Push Relabel Algorithm
(Goldberg & Tarjan)

FTP approach: Maintain valid flows and keep increasing
until t is not reachable from s in residual network.

Push relabeled approach: Maintain t is not reachable
from s in residual network and keep changing
flows until t becomes valid.

Precond: \( f: \{x,y\} \to \mathbb{R}_+ \)

1. \( f(x,y) \leq u(x,y) \)
2. \( \sum_{x \in V \setminus \{s,t\}} f(x,y) - f(y,x) \geq 0 \)

A precond where excess flow at all vertices
\( y \in V \setminus \{s,t\} \) is 0 is a valid flow.

Residual network for a precond defined as before:

- \( u_g(x,y) = u(x,y) - f(x,y) \)
- \( u_g(y,x) = u(y,x) + f(x,y) \)

Push: If \( \text{excess at } y \geq r \), and \( u_g(y,w) \geq r \),
send precond of \( r \) on edge \( (y,w) \) in residual
network = compute precond on residual network
with precond = residual graph as for flows.

Algorithm

Initailize:
- saturate all edges out of \( s \) with precond
- \( h(s) = |V| \)

Repeat until precond is feasible:
- Push excess flow on edge along
  decreasing height
- Relabel vertex with excess flow by
  increasing height by 1

Output feasible flow

Lemma 1: For every vertex \( x \), \( h(x) \leq 2|V| - 1 \)
\( \Rightarrow \) # of relabeled operations \( \leq 2|V|^2 \)

Lemma 2: # of saturating pushes on an edge \( \leq |V| \)
\( \Rightarrow \) # of saturating pushes \( \leq 2|E||V| \)

Lemma 3: # of non-saturating pushes \( \leq 4|V|^2 |E| \)

Lemma 4: All push/relabeled operations take \( O(1) \) time

Theorem: Time complexity is \( O(|V|^3 |E|) \).

Proof of lemma 1:

Lemma: If a vertex \( y \) has excess flow \( > 0 \), then \( y \)
\( \in \) path in residual network.

Proof:

[Diagram showing flow and residual network]
Lemma: If \( u_x (x, y) > 0 \), then \( h(x) > h(y) \).

Proof: Initially, edges into \( s \) go uphill; all other edges are flat.

Path \( x \rightarrow y \), new \( (y, x) \) edge goes uphill.

Relabel \( h'(x) = h(x) + 1 \).

\[ h'(y) = h(y) \geq h(x) = h(x) - 1 \] (applying after the increase in height)

Proof of Lemma 1: If the height of a node is increased, it has excess flow > 0.

\[ f \] \((u, v)\) path in residual network.

\[ h(v) \leq h(s) + \text{length of path} \leq |V| + |V| - 1 = 2|V| - 1 \]

Proof of Lemma 2: There must be \( s \) relabels on \( x \) between two saturating pushes on \((x, y)\) since there must be a push on \((y, x)\) in between.

\[ \# \text{ of saturating pushes on } (x, y) \leq |V| \]

Proof of Lemma 3:

Potential \( \phi = \sum h(v) \).

\( \phi_{init} = 0 \)

\( \phi_{final} = 0 \) (only \( t \) has excess, but \( h(t) = 0 \))

Relabel: \( \Delta \phi = 1 \)

Saturating push: \( \Delta \phi \leq 2|V| \) (by creating on \((x, y)\) and \((y, x)\))

Non-saturating push: \( \Delta \phi \leq h(y) - h(x) \)
\[
\begin{align*}
\text{# of non-contradicting pushes} & \leq (2|V|-1)(|V|-1) \\
& + 2|V| (2|V||E|) \\
& \leq 4|V|^2 |E|
\end{align*}
\]

We have shown that total # of ops

\[= O(|V|^2 |E|)\]

Push on one of max height \[= O(|V|^2 |E|)\]

Dynamic tree with simultaneous pushes

\[= O(|V||E| \log |V|)\]

Best strongly polynomial also!