Monte Carlo Algorithms

Algorithm "succeeds" w.h.p. (usually $1 - \frac{1}{n^2}$) for any constant $n > 0$.

**Contraction Algorithm for Global Min cut**

- $G_0 = G$
- for $i = n-1$ to 2 do
  - contract an edge chosen uniformly at random in $G_{i+1}$ to produce $G_i$
  - remove self-loops if any

Output cut represented by two supervertices in $G_2$

**Analysis**

**Lemma:** For any given global mincut $(S, \overline{S})$,

$$P[(S, \overline{S}) \text{ is output}] = \Omega(1/n^2)$$

**Proof:**

$$P[(S, \overline{S}) \text{ is a cut in } G_i \mid (S, \overline{S}) \text{ is a cut in } G_{i+1}] \geq 1 - \frac{\Delta}{(i+1)(\Delta/2)}$$

where deg. of any vertex $\geq \Delta$

$$= \frac{i-1}{i+1} = \prod_{i=n-1}^{2} \frac{i-1}{i+1} = \frac{1}{n(n-1)} = \Omega(1/n^2)$$

How do we obtain a high probability bound?
- Repeat $\Omega(n \log n)$ times and output the smallest cut.

**Running time:** The contraction algorithm can be implemented in $\tilde{\Omega}(n^4)$ time.
Improving to an $\tilde{O}(n^2)$ algorithm:

Intuition: prob. of success decreases when the number of vertices is small.

Computation Tree

How many independent trials for $G_i$? Call this $N_i$.
Recall: \[ P\left[(s, t) \text{ survives in } G_i\right] = \Theta\left(\frac{n^2}{i^2}\right) \]
\[ P\left[(s, t) \text{ survives in } G_i \text{ for some trial}\right] = 1 - \left(1 - \frac{n^2}{i^2}\right)^{N_i} \]
Set $N_i = \left(\frac{n^2}{i^2}\right) \log n$.

What is the size of the computation tree?
\[ \sum_{i=n}^{2} N_i = O\left(n^2 \log n\right) \]

Why is this not formally correct? $N_i$ iterations NOT independent!
Formally, run contraction algorithm till $i = \sqrt{n}$, spawn two copies $G_1$ and $G_2$ and return mincut among those reported recursively by $G_1$ and $G_2$.

\[ T(n) = 2T\left(n/\sqrt{2}\right) + O(n^2) \Rightarrow T(n) = O\left(n^2 \log n\right) \]

Prob of success $\geq \frac{1}{\log n}$

Proof: recursion. Time of $d = \log n$ depth binary tree

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What is the probability of a path of all heads?

\[ P_d = \frac{1}{2} \left( 1 - \left( 1 - P_{d-1} \right)^2 \right) \]

Max, \( \frac{1}{2} \left( 1 - \left( 1 - \frac{1}{d-1} \right)^2 \right) \)

\[ \geq \frac{1}{2} \cdot \frac{2(d-2)}{(d-1)^2} \geq \frac{1}{d} \]

Prob of success = \( 1/\log n \)