CPS 230
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Design and Analysis of Algorithms
Fall 2001

Homework 2 (due before class, Monday, September 24, 2001

1 Recurrences, Master Method, Sorting [Chapter 4 in CLRS]

1. [CLRS 4.2-5] Solve the recurrence $T(n) = T(\alpha n) + T((1 - \alpha)n) + n$, where $0 < \alpha < 1$ constant. (Hint: Use a recursion tree. Use the parsetree.sty package to \LaTeX{} the tree.)

2. Give asymptotic upper and lower bounds using $\Theta$ notation for $T(n)$ at those points where $T(n)$ is defined, for the following recurrences. Assume $T(n)$ is constant for $n \leq 20$. Make your bounds as tight as possible, and justify your answers.

(a) $T(n) = 2T(n/3) + n^3$
(b) $T(n) = 3T(n/4) + \sqrt{n}$
(c) $T(n) = 9T(n/4) + n^2$
(d) $T(n) = T(n - 6) + n$
(e) $T(n) = T(\sqrt{n}) + 1$
(f) $T(n) = 2T(n/2) + n \lg^2 n$
(g) $T(n) = 3T(n/3 + 5) + n/2$
(h) $T(n) = 2T(n/2) + n/\lg n$
(i) $T(n) = T(n - 2) + 1/n$
(j) $T(n) = \sqrt{n}T(\sqrt{n}) + n$

2 Quicksort, Median, Matrices [Chapter 7, 9, 28 in CLRS]

3. [CLRS 9.3-1] In the algorithm Select, the input elements are divided into groups of 5. Will the algorithm work in linear time if they are divided into groups of 7? How about groups of 3?

4. [CLRS 9.2] For $n$ distinct elements $x_1, x_2, \ldots, x_n$ with positive weights $w_1, w_2, \ldots, w_n$ such that $\sum_{i=0}^{n} w_i = 1$, the weighted (lower) median is the element $x_k$ satisfying

$$\sum_{x_i < x_k} w_i < \frac{1}{2} \text{ and } \sum_{x_i > x_k} w_i \leq \frac{1}{2}.$$
(a) Argue that the median of \( x_1, x_2, \ldots, x_n \) is the weighted median of the \( x_i \) with weights \( w_i = 1/n \) for \( i = 1, 2, \ldots, n \).

(b) Show how to compute the weighted median of \( n \) elements in \( O(n \log n) \) worst-case time using sorting.

(c) Show how to compute the weighted median in \( \Theta(n) \) worst-case time using a linear-time median algorithm.

The post office location problem is defined as follows. We are given \( n \) points \( p_1, p_2, \ldots, p_n \) with associated weights \( w_1, w_2, \ldots, w_n \). We wish to find a point \( p \) (not necessarily one of the input points) that minimizes the sum \( \sum_{i=1}^{n} w_i d(p, p_i) \), where \( d(a, b) \) is the distance between points \( a \) and \( b \). Intuitively, the point \( p \) represents a location of the post office that minimizes average distance.

(d) Argue that the weighted median is a best solution for the 1-dimensional post office location problem, in which points are simply real numbers and the distance between points \( a \) and \( b \) is \( d(a, b) = |a - b| \).

(e) Find the best solution for the 2-dimensional post office location problem, in which the points are \((x, y)\) coordinate pairs and the distance between points \( a = (x_1, y_1) \) and \( b = (x_2, y_2) \) is the Manhattan distance given by \( d(a, b) = |x_1 - x_2| + |y_1 - y_2| \).

5. Show that the smallest and the second smallest of \( n \) distinct elements can be found with \( n + \lfloor \log n \rfloor - 2 \) comparisons in the worst case.