Seventh Homework Assignment

The purpose of this assignment is to help you prepare for the final exam. Solutions will neither be graded nor even collected.

Problem 1. (20 = 5 + 15 points). Consider the class of satisfiable boolean formulas in conjunctive normal form in which each clause contains two literals, \(2\text{-SAT} = \{ \varphi \in \text{SAT} \mid \varphi \text{ is 2-CNF} \} \).

(a) Is \(2\text{-SAT} \in \text{NP}\)?
(b) Is there a polynomial-time algorithm for deciding whether or not a boolean formula in 2-CNF is satisfiable? If your answer is yes, then describe and analyze your algorithm. If your answer is no, then show that \(2\text{-SAT} \in \text{NP}-\text{C}\).

Problem 2. (20 points). Let \(A\) be a finite set and \(f\) a function that maps every \(a \in A\) to a positive integer \(f(a)\). The \textsc{Partition} problem asks whether or not there is a subset \(B \subseteq A\) such that

\[
\sum_{b \in B} f(b) = \sum_{a \in A - B} f(a).
\]

We have learned that the \textsc{Partition} problem is \text{NP}-complete. Given positive integers \(j\) and \(k\), the \textsc{Sum of Squares} problem asks whether or not \(A\) can be partitioned into \(j\) disjoint subsets, \(A = B_1 \cup B_2 \cup \ldots \cup B_j\), such that

\[
\sum_{i=1}^{j} \left( \sum_{a \in B_i} f(a) \right)^2 \leq k.
\]

Prove that the \textsc{Sum of Squares} problem is \text{NP}-complete.

Problem 3. (20 = 10+10 points). Let \(G\) be an undirected graph. A path in \(G\) is \emph{simple} if it contains each vertex at most once. Specifying two vertices \(u, v\) and a positive integer \(k\), the \textsc{Longest Path} problem asks whether or not there is a simple path connecting \(u\) and \(v\) whose length is \(k\) or longer.

(a) Give a polynomial-time algorithm for the \textsc{Longest Path} problem or show that it is \text{NP}-hard.
(b) Revisit (a) under the assumption that \(G\) is directed and acyclic.

Problem 4. (20 = 10 + 10 points). Let \(A \subseteq 2^V\) be an abstract simplicial complex over the finite set \(V\) and let \(k\) be a positive integer.

(a) Is it \text{NP}-hard to decide whether \(A\) has \(k\) or more disjoint simplices?
(b) Is it \text{NP}-hard to decide whether \(A\) has \(k\) or fewer simplices whose union is \(V\)?

Problem 5. (20 points). Let \(G = (V, E)\) be an undirected, bipartite graph and recall that there is a polynomial-time algorithm for constructing a maximum matching. We are interested in computing a minimum set of matchings such that every edge of the graph is a member of at least one of the selected matchings. Give a polynomial-time algorithm constructing an \(O(\log n)\) approximation for this problem.