Problem 1
Consider the following problem. Given an undirected graph $G = (V, E)$ with a non-negative weight function $w(\cdot)$ on the vertices $V$, a set of terminals $T \subseteq V$ and a number $K$, decide whether there exists a subgraph $G'$ of $G$ such that in $G'$, there exists a path between every pair of terminals and the total weight of nodes in $G'$ is at most $K$. Show that this problem is NP-Complete. (Hint: Reduce from the set cover problem.)

Problem 2
In the optimization version of the above problem, the objective is to find such a subgraph $G'$ with minimum total node weight. First show that you can assume, without loss of generality, that all terminals are leaves of weight 0 in $G$.

We now describe a greedy algorithm for this problem. Define a spider to be a tree such that at most one vertex has degree greater than 2.

i. Initially, let $G_0$ be the set terminals $T$ with no edge.

ii. At the $i$-th step, choose a subgraph $S$ of $G$ such that

- $S$ is a spider
- all leaves of $S$ are terminals
- $S$ has the minimum cost-benefit ratio $\frac{C(S)}{B(S)}$. Here the cost $C(S)$ is the total node weight of $S$ and the benefit $B(S) = \alpha(G_i) - \alpha(G_i \cup S)$, where $\alpha(G)$ denotes the number of connected components in graph $G$.

Let $G_{i+1} = G_i \cup S$. If $G_{i+1}$ connects all terminals, stop the algorithm and output $G_{i+1}$; else go to ii.

Answer the following questions about this greedy algorithm.

(1) Give a polynomial time implementation of this greedy algorithm.

(2) Show that this algorithm returns an $O(\log \kappa)$-approximate solution, where $\kappa = |T|$. (Hint: Think of the analysis of the greedy algorithm for the set cover algorithm.)