- Algorithm is always correct but running time bounds hold in expectation.
  - e.g. Randomized Quicksort
    - review quicksort
    - worst case: \( T(n) = 2T(n-1) + O(1) = O(n^2) \)

In randomized quicksort, pick pivot uniformly at random in each subproblem.

**Analysis 1 (Backward Analysis)**

In step \( k \), # of pivots increases from \( k-1 \) to \( k \) by selecting a random pivot from the elements not selected yet. Looking backwards, in step \( k \) (steps are still indexed from the beginning), the # of pivots decreases from \( k \) to \( k-1 \).

Claim: Given a set of \( k \) pivots at the end of step \( k \), the pivot that was selected in the \( k \)th step is uniform distributed among these \( k \) pivots.

Proof: for any two elements, their relative order of being selected as pivot is uniform.

Lemmas: The expected cost in step \( k \) is \( \leq \frac{2(n-k)}{k} \).

Proof: The sum of costs over all the \( k \) pivots being the last one is \( \leq 2n \).

Cor: The expected cost of randomized quicksort is \( O(n \log n) \).
Consider the comparison that an element $x_i$ is part of. If such a comparison splits the subproblem in $(\frac{3}{4}, \frac{1}{4})$ or a more balanced ratio, call it a "good" comparison; prob of good comparison:

**Fact:** $x_i$ is in $\leq \log_{1/3} n$ good comparisons.

**Lemma:** $\Pr(\text{# of comparisons for } x_i \text{ till you get } k \text{ good comparisons} > (1+\epsilon) x)$ $\leq e^{-\epsilon^2 x}.$

**Proof:** Since comparisons being good or bad are independent, we use Chernoff bounds.

Choose $\epsilon = \sqrt{\ln (1/3)}$ and $k = \log_{1/3} n.$

$\Pr(\text{# of comparisons for } x_i > (1+\sqrt{\ln (1/3)}) \log_{1/3} n)$ $\leq \frac{1}{n^3}.$

$\Pr(\exists x_i \text{ s.t. } \# \text{ of comparisons for } x_i = O(\log n) \leq \frac{1}{n^3}.$

So running time $= O(n \log n)$ $\geq 1 - \frac{1}{n^3}.$