Diffusion Equations

DISCRETE

\[ P(t_0 + \Delta t, x_0) = \frac{1}{2} P(t_0, x_0 - \Delta x) + \frac{1}{2} P(t_0, x_0 + \Delta x) \]
\[ P(t_0 + \Delta t, x_0) = \frac{1}{3} P(t_0, x_0 - \Delta x) + \frac{1}{3} P(t_0, x_0) + \frac{1}{3} P(t_0, x_0 + \Delta x) \]
\[ P(t_0 + \Delta t, x_0, y_0) = \frac{1}{4} P(t_0, x_0 - \Delta x, y_0 - \Delta y) + \frac{1}{4} P(t_0, x_0 + \Delta x, y_0 - \Delta y) + \frac{1}{4} P(t_0, x_0 - \Delta x, y_0 + \Delta y) + \frac{1}{4} P(t_0, x_0 + \Delta x, y_0 + \Delta y) \]

Each of the above can be described by a grid/lattice and a stencil. The stencils specify the rules of local moves or interactions at each temporal step, and the grid gives a global map.

Various interpretations:

- random walk: the possibility at location \( x_0 \) at time \( t_0 + \Delta t \)
- Markov chains
- an equilibrium process
- a smoothing process
- a viscosity process
- not limited in physical science fields
CONTINUOUS

\[ \frac{\partial P(t,x)}{\partial t} = \gamma \frac{\partial^2 P(t,x)}{\partial x^2} \]

\[ \frac{\partial P(t,x,y)}{\partial t} = \gamma \left\{ \frac{\partial^2 P(t,x,y)}{\partial x^2} + \frac{\partial^2 P(t,x,y)}{\partial y^2} \right\} \]

\[ \frac{\partial P(t,r)}{\partial t} = \gamma \text{div}(\nabla P(r)) = \rho \Delta P(r), \quad r \in \mathbb{R}^d \]

- a typical type of transporting phenomena
  in continuum quantities
  in large ensembles: thermal diffusion, electron diffusion, molecular diffusion, photon diffusion

- temporal and spatial aspects in transporting

- boundary conditions besides initial conditions

By the study of the connections between the discrete and continuous models, observe the relation between temporal sampling and spatial sampling

\[ \Delta t = c \cdot \Delta^2 x \]

- This condition says the temporal sampling shall be denser than the spatial sampling

- In many situations this condition is not too hard to meet, i.e., when the magnitude of \( c \) is large enough or reciprocally proportional to \( \Delta x \)