Consider the following scoring rules in a binary setting (the outcome is 1 or 0, and $p$ indicates the probability of 1):

1. 
   \[ S(p, 1) = \frac{\sqrt{p}}{2} + \frac{1}{2\sqrt{p}} \]
   \[ S(p, 0) = \frac{\sqrt{p}}{2} \]

2. 
   \[ S(p, 1) = \begin{cases} 
   1 & \text{if } p > 1/2 \\
   0 & \text{if } p = 1/2 \\
   -1 & \text{if } p < 1/2
   \end{cases} \]
   \[ S(p, 0) = \begin{cases} 
   -1 & \text{if } p > 1/2 \\
   0 & \text{if } p = 1/2 \\
   1 & \text{if } p < 1/2
   \end{cases} \]

3. 
   \[ S(p, 1) = -2p^3 + 3p^2 \]
   \[ S(p, 0) = -2p^3 \]

4. 
   \[ S(p, 1) = p^2 - 2p + 5/4 \]
   \[ S(p, 0) = p^2 - 2p + 1/4 \]

Answer the following questions for each of these rules, with proof / explanation in each case.

(a) Is the rule strictly proper, proper (but not strictly), or not proper? You are encouraged to do this in two different ways.

(b) Suppose the principal prefers outcome 1 over outcome 0 (and, hence, prefers larger values of $p$). Which of the rules are both proper and principal-aligned?

(c) For each of the rules, give the maximum amount that the principal might pay if she uses this in a market scoring rule with initial probability $1/2$. 
