Maps

COMPSCI 230 — Discrete Math

February 2, 2017
Outline

1 Maps and Functions
Maps and Functions

- Maps and functions describe *associations*
- Most things that do something can be viewed as maps or functions
- Programs are either maps or functions, and associate inputs to outputs
- A map may be *non-deterministic*: same input $\rightarrow$ different outputs
- The Google search engine is a nondeterministic map from queries to lists of results
- A function is a *deterministic* map: same input $\rightarrow$ same output
- Sphere volume is a function of radius
- ...
Maps and Functions

• Maps are so general that it is hard to say anything about them.
• Functions are constrained maps, and even more constraints lead to mathematically interesting objects.
• Cool application: the size of infinite sets (Georg Cantor, 1845-1918)
• The set of reals and the power set of the naturals have equal size.

Get to use fancy typefaces: There are $\aleph_0$ naturals and $\mathfrak{c}$ reals.
• “Aleph-zero” is arabic ’A’ and “cee” is German Fraktur ’c’.
• Get to prove that $\aleph_0 - 1 = \aleph_0$. This is good for parties:

\[
\begin{align*}
\aleph_0 & \text{ bottles of beer on the wall,} \\
\aleph_0 & \text{ bottles of beer.} \\
& \text{Take one down, and pass it around,} \\
\aleph_0 & \text{ bottles of beer on the wall} \\
& (\text{repeat})
\end{align*}
\]

Maps

A map

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>p</td>
</tr>
<tr>
<td>b</td>
<td>p</td>
</tr>
<tr>
<td>c</td>
<td>q</td>
</tr>
<tr>
<td>c</td>
<td>r</td>
</tr>
<tr>
<td>d</td>
<td>?</td>
</tr>
<tr>
<td>?</td>
<td>s</td>
</tr>
</tbody>
</table>

- **Domain** \( D \)
- **Codomain** \( C \)
- Some \( x \)'s map to some \( y \)'s
- There may be both gaps and duplicates in either column
- \( \exists x \in D, \exists y \in C : y = f(x) \)
A map is exactly $D$, $C$, and a nonempty subset of $D \times C$

Two problems prevent this map from being a function:

- Some $x \in D = \{a, b, c, d\}$ have no $f(x)$
- Some $x \in D$ have more than one $f(x)$
- OK that some $y$ is not matched
- OK that different $x$s are matched to the same $y$
Functions

A function from $D$ into $C$

$D = \{a, b, c\}$, $C = \{p, q, r, s\}$

- A function assigns exactly one $y \in C$ to every $x \in D$
- All $x \in D$ show up in the first column exactly once
- $\forall x \in D, \exists y \in C : y = f(x)$ (at least one) and
  $\forall x, x' \in D, y, y' \in C : y = f(x), y' = f(x'), y \neq y' \Rightarrow x \neq x'$ (at most one)
- All functions are “into”
Functions and Injections

A function but not an injection

\[ D = \{a, b, c\}, \quad C = \{p, q, r, s\}\]

- Some \( y \in C \) are associated to more than one \( x \in D \)
Injections

An injection

<table>
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</tr>
</tbody>
</table>

- $D = \{a, b, c\}$, $C = \{p, q, r, s\}$
- An injection is a function that assigns a different $y \in C$ to every $x \in D$
- Function, plus distinct $x$'s in the first column match distinct $y$'s in the second
- Function, plus $\forall x, x' \in D : x \neq x' \Rightarrow f(x) \neq f(x')$
- “One-to-one function” is synonym for “injection”
Functions and Surjections

A function but not a surjection

\[ D = \{a, b, c\}, \quad C = \{p, q, r\} \]

• Some \( y \in C \) are unassigned
Surjections

A surjection

$D = \{a, b, c\}, C = \{p, q\}$

A surjection is a function that assigns every $y$ to some $x$

Function, plus every $y \in C$ shows up in the second column

Function, plus $\forall y \in C, \exists x \in D : y = f(x)$

“Onto function” is synonym for “surjection”
An injection but not a bijection

- $D = \{a, b, c\}$, $C = \{p, q, r, s\}$
- Function and one-to-one, but some $y \in C$ are unassigned
- Some $y$s in the second column have no matching $x$
Bijections

A bijection

• $D = \{a, b, c\}, \ C = \{p, q, r\}$

• A bijection has exactly one $y \in C$ for every $x \in D$ and *vice versa*

• All $x \in D$ show up in the first column exactly once, all $y \in C$ show up in the second column exactly once, no gaps

• Bijection = injection $\cap$ surjection

• “One-to-one and onto function” is synonym for “bijection”
Bijections are Invertible

- \( D = \{a, b, c\} \), \( C = \{p, q, r\} \)
- Bijection = injection \( \cap \) surjection
- Bijection = one-to-one \( \cap \) onto
- Bijection = injection both ways
- \(|D| = |C|\) (same cardinality)
- This is where Cantor comes in...
$D \subseteq \mathbb{R}, C \subseteq \mathbb{R}, \text{Continuous } f$

- **Map**
- **Function**
- **Injection**
- **Surjection**
- **Bijection**