Graphs

COMPSCI 230 — Discrete Math

March 23, 2017
Overview

1. Graphs
2. Graph Nomenclature
3. Graph Examples
4. The Adjacency Matrix
5. Tours
6. Euler and Hamilton Circuits
(Undirected) Graphs

- **Graphs** are items together with binary relations
- Items are called *vertices* (or nodes)
- Relations are called *edges* (or arcs)
- Formally:
  - \( G = (V, E) \)
  - \( G \) is an ordered pair
  - \( V \) is a set of vertices
  - Every element of \( E \) is an *unordered* pair (a set) from \( V \times V \)
  - Note the two distinct edges between \( c \) and \( d \)

\[ G = (V, E) \]
\[ V = \{a, b, c, d\} \]
\[ E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{c, d\}, \{d, d\}\} \]
Vertices and Edges

- The two copies of \{c, d\} are parallel edges
- \{d, d\} is a loop
- a and b are adjacent to each other (neighbors of each other)
- \{a, b\} is incident to a and b
- The degree of a vertex is the number of edge-ends incident to it: deg(a) = 3; deg(b) = 2; deg(c) = 4; deg(d) = 5
- A graph is simple if it has no loops and no parallel edges
Paths

- $G = (V, E) = \{a, b, c, d, e\}, \{e_1, \ldots, e_{10}\}$ is a graph
- The sequence $(a, e_2, c, e_8, c, e_4, b, e_6, e, e_6, b)$ is a path from $a$ to $b$
- $(a, e_2, c, e_4, b, e_6, e)$ is a simple path from $a$ to $e$ (no repeated vertices)
- $(a, e_1, b, e_6, e)$ is a shortest path from $a$ to $e$
- $(a, e_3, d, e_5, b, e_7, e, e_{10}, d, e_3, a)$ is a closed path (first = last)
- $(a, e_2, c, e_8, c, e_9, d, e_3, a)$ is a circuit (non-empty closed path with distinct edges)
- $(a, e_2, c, e_4, b, e_5, d, e_3, a)$ is a polygon (connected graph with only degree-2 vertices; degree counted in the path)
- A polygon is a circuit, but not vice versa
- A graph is connected if there is a path between any two of its vertices
Subgraphs

- \((\{a, b, d, e\}, \{e_1, e_3, e_5, e_6\})\) is a subgraph of \(G\) (a graph made of vertices and edges from \(G\))
- \((\{a, b, c, d, e\}, \{e_1, e_3, e_5, e_6, e_9\})\) is a spanning subgraph of \(G\), because it covers all the vertices
- \((\{a, b, c, d, e\}, \{e_3, e_5, e_6, e_9\})\) is a spanning tree of \(G\): a spanning subgraph that is also a tree
  [A tree is a graph with no polygons]
- Spanning trees arise in graph traversals
Where is the Graph?
Graphs Abstract Away Geometry

- Shady Grove
- Metro Center
- Rosslyn
- Pentagon
- Vienna
- Franconia
- Glenmont
- Mount Vernon
- Chinatown
- L’Enfant Plaza
- King Street
- Huntington
- Greenbelt
- Fort Totten
- New Carrolton
- Stadium
- Largo Town
- Branch Avenue
Transfer Stations Only?

Shows which transfer station connects to which transfer station
Can reason about interconnectivity and redundancy
The Adjacency Matrix

<table>
<thead>
<tr>
<th></th>
<th>Metro Center</th>
<th>Rosslyn</th>
<th>Pentagon</th>
<th>Mount Vernon</th>
<th>Chinatown</th>
<th>L’Enfant Plaza</th>
<th>King Street</th>
<th>Fort Totten</th>
<th>Stadium</th>
</tr>
</thead>
<tbody>
<tr>
<td>Metro Center</td>
<td>2</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rosslyn</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pentagon</td>
<td></td>
<td>1</td>
<td></td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mount Vernon</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Chinatown</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L’Enfant Plaza</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>King Street</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fort Totten</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stadium</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Degree**

|         | 5 | 3 | 4 | 3 | 6 | 7 | 2 | 2 | 2 |

- Number of edges that connect any two entries
- Entries not shown are 0
- The adjacency matrix is symmetric for undirected graphs
- A simple graph has a binary adjacency matrix with zero diagonal
- The sum of each column (or row) is the degree of the corresponding node
- Sum of degrees $= 34 = $ twice the number of edges
Could label edges by which metro line connects any two transfer stations
Every color gives an open path on this graph
Which Transfer Station Connects Which Lines?

- What is a vertex or an edge is up to us
- The graph is in the eye of the beholder
- **Warning:** It does not always make sense to swap vertices with edges
Tours

• The path 
  \((e, e_7, b, e_1, a, e_2, c, e_5, b, e_5, c, e_9, d, e_4, a, e_2, c, e_8, c, e_3, a, e_1, b)\)
  is an edge tour of \(G\), because it traverses every edge.

• It is also a vertex tour, because it traverses every vertex.

• The path \((e, e_7, b, e_5, c, e_9, d, e_4, a, e_2, c, e_8, c, e_3, a, e_1, b, e_6, d)\) is
  an Euler tour of \(G\), because it traverses every edge exactly once.

• The path \((a, e_2, c, e_9, d, e_6, b, e_7, e)\) is a Hamilton tour because it
  traverses every vertex exactly once.
Euler and Hamilton Circuits

The circuit 
\((d, e_{10}, e, e_7, b, e_5, c, e_9, d, e_4, a, e_2, c, e_8, c, e_3, a, e_1, b, e_6, d)\)

is an Euler circuit of \(G\), because it traverses every edge exactly once.

The circuit 
\((a, e_2, c, e_9, d, e_{10}, e, e_7, b, e_1, a)\)

is a Hamilton circuit because it traverses every vertex exactly once (before returning to the original vertex).
Euler’s Theorem

• Not every graph has an Euler tour, an Euler circuit, a Hamilton tour, or a Hamilton circuit
• This simple graph has none of them:

  ![](image_url)

  • Euler’s theorems: For a connected graph $G$,
    • All nodes of even degree $\leftrightarrow G$ has an Euler circuit (which is also an Euler tour)
    • Exactly two nodes of odd degree $\leftrightarrow G$ has an open Euler tour
    • Tour starts at one odd-degree node and ends at the other
    • No similarly clean characterization of Hamiltonian graphs is known
Euler Circuit Example

- Domino tiles:
  
  ![Domino Tiles Diagram]

- Place them in sequence so that adjacent numbers match:
  
  ![Sequence of Dominoes Diagram]

- Can we make a loop that uses all the 28 tiles?
The Domino Graph

- Where is the graph?
- Vertices are the dot numbers, edges are the tiles

- A loop that uses all the 28 tiles is an Euler circuit
- Simple algorithms exist to find Euler circuits
- In this case, it is easy to draw one by hand
A Domino Euler Circuit
Hamilton Circuit Example

• In chess, a single move by the knight is by one cell in one direction and two in the other

• The knight’s graph is the obvious one

• A *re-entrant knight’s tour* is a sequence of moves that takes the knight to each square exactly once, with the first and last cell a knight’s move apart
A Knight’s Hamilton Circuit

- First discovered by al-Adli in 840 A.D.
- This is a Hamilton circuit

[Image from http://www.dharwadker.org/pirzada/applications/]