Exercises

The credit assignment reflects a subjective assessment of difficulty. A typical question can be answered using knowledge of the material combined with some thought and analysis.

1. **Hessian** (two credits). Compute the Hessian and, if defined, the index of the origin, which is critical for each function in the list below.

   (i) \( f(x_1, x_2) = x_1^2 + x_2^2. \)
   
   (ii) \( f(x_1, x_2) = x_1x_2. \)
   
   (iii) \( f(x_1, x_2) = (x_1 + x_2)^2. \)
   
   (iv) \( f(x_1, x_2, x_3) = x_1x_2x_3. \)
   
   (v) \( f(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3. \)
   
   (vi) \( f(x_1, x_2, x_3) = (x_1 + x_2 + x_3)^2. \)

2. **Approximate Morse function** (two credits). Let \( M \) be a geometrically perfect torus in \( \mathbb{R}^3 \), that is, \( M \) is swept out by a circle rotating about a line that lies in the same plane but does not intersect the circle. Let \( f : M \to \mathbb{R} \) measure height parallel to the symmetry axis and note that \( f \) is not Morse.

   (i) Describe a Morse function \( g : M \to \mathbb{R} \) that differs from \( f \) by an arbitrarily small amount, \( \|f - g\|_\infty < \varepsilon. \)

   (ii) Draw the Reeb graphs of both functions.

3. **Morse-Smale complex** (two credits). Let \( M \) be the torus in Question 2 and let \( f : M \to \mathbb{R} \) measure height along a direction that is almost but not quite parallel to the symmetry axis of the torus.

   (i) Draw the Morse-Smale complex of the height function.

   (ii) Give the chain, cycle, boundary groups defined by Floer homology.

4. **Quadrangles** (three credits). Let \( M \) be a 2-manifold and \( f : M \to \mathbb{R} \) a Morse-Smale function.

   (i) Prove that each 2-dimensional cell of the Morse-Smale complex of \( f \) is a quadrangle. In other words, each 2-dimensional cell is an open disk whose boundary can be decomposed into four arcs each glued to an edge in the complex.

   (ii) Draw a case in which one edge is repeated so that the disk is glued to only three edges but twice to one of the three.
5. **Distance from a point** (three credits). Let \( M \) be the torus swept out by a unit circle rotating at unit distance from the \( x_3 \)-axis. More formally, \( M \) consists of all solutions to \( x_1^2 + x_2^2 = (2 \pm \sqrt{1-x_3^2})^2 \) in \( \mathbb{R}^3 \). For a point \( z \in \mathbb{R}^3 \) consider the function \( f_z : M \to \mathbb{R} \) defined by \( f_z(x) = ||x - z|| \).

(i) Describe the set of points \( z \) for which \( f_z \) violates property (i) of a Morse function.

(ii) Describe the set of points \( z \) for which \( f_z \) is not a Morse function.

6. **Non-simple PL critical point** (one credit). Let \( K \) be a triangulation of a 3-manifold and \( f : K \to \mathbb{R} \) a generic PL function.

(i) Assuming \( f \) is a PL Morse function, draw the lower links of the four types of simple PL critical points that can occur.

(ii) Assuming \( f \) is not a PL Morse function, draw the lower link of a non-simple PL critical point.

7. **Lower and upper star filtrations** (one credit). Let \( K \) be a simplicial complex, \( f : K \to \mathbb{R} \) a generic PL function, and \( f(u_1) < f(u_2) < \ldots < f(u_n) \) the ordering of the vertices by function value. For \( 0 \leq i \leq n \) let \( K_i \) be the union of lower stars of the first \( i \) vertices and let \( K^i \) be the union of upper stars of the last \( n - i \) vertices. Let \( f(u_i) < t < f(u_{i+1}) \).

(i) Prove that the sublevel set for threshold \( t \), \( f^{-1}(-\infty, t] \), has the same homotopy type as \( K_i \).

(ii) Prove that the superlevel set for threshold \( t \), \( f^{-1}[t, \infty) \), has the same homotopy type as \( K^i \).

8. **Morse inequalities** (two credits). Recall that the unstable manifolds of a Morse function \( f : \mathbb{M} \to \mathbb{R} \) are the stable manifolds of \(-f\). Furthermore, if \( \mathbb{M} \) is a \( d \)-manifold then an index \( p \) critical point of \( f \) is an index \( d - p \) critical point of \(-f\).

(i) Use this symmetry to formulate collections of inequalities symmetric to the weak and strong Morse inequalities of \( f \).

(ii) Use these inequalities to prove that the Euler characteristic of \( \mathbb{M} \) vanishes if \( d \) is odd.