18 Surfaces

Graphs may be drawn in two, three, or higher dimensions, but they are still intrinsically only 1-dimensional. One step up in dimensions, we find surfaces, which are 2-dimensional.

Topological 2-manifolds. The simplest kind of surfaces are the ones that on a small scale look like the real plane. A space $\mathbb{M}$ is a 2-*manifold* if every point $x \in \mathbb{M}$ is locally homeomorphic to $\mathbb{R}^2$. Specifically, there is an open neighborhood N of x and a continuous bijection $h : N \to \mathbb{R}^2$ whose inverse is also continuous. Such a bicontinuous map is called a *homeomorphism*. Examples of 2-manifolds are the open disk and the sphere. The former is not compact because it has covers that do not have finite subcovers. Figure 79 shows examples of compact 2-manifolds. If we add the boundary circle to the open disk

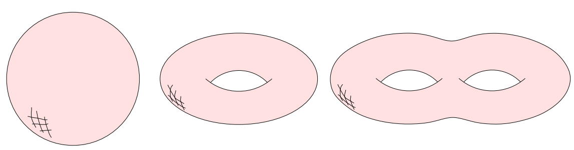

Figure 79: Three compact 2-manifolds, the open sphere, the torus, and the double torus.

we get a closed disk which is compact but not every point is locally homeomorphic to $\mathbb{R}^2$. Specifically, a point on the circle has an open neighborhood homeomorphic to the closed half-plane, $\mathbb{H}^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0\}$. A space whose points have open neighborhoods homeomorphic to $\mathbb{R}^2$ or $\mathbb{H}^2$ is called a 2-*manifolds with boundary*; see Figure 80 for examples. The *boundary* is the subset

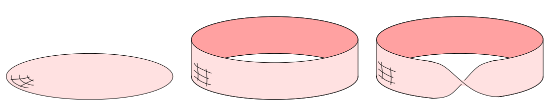

Figure 80: Three 2-manifolds with boundary, the closed disk, the cylinder, and the Möbius strip.

of points with neighborhoods homeomorphic to $\mathbb{H}^2$. It is a 1-manifold (without boundary), that is, every point is locally homeomorphic to $\mathbb{R}$. There is only one type of compact, connected 1-manifold, namely the closed curve. In topology, we do not distinguish spaces that are homeomorphic to each other. Hence, every closed curve is like every other one and they are all homeomorphic to the unit circle, $\mathbb{S}^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$.

Triangulations. A standard representation of a compact 2-manifold uses triangles that are glued to each other along shared edges and vertices. A collection K of triangles, edges, and vertices is a *triangulation* of a compact 2-manifold if

I. for every triangle in K, its three edges belong to K, and for every edge in K, its two endpoints are vertices in K;

II. every edge belongs to exactly two triangles and every vertex belongs to a single ring of triangles.

An example is shown in Figure 81. To simplify language, we call each element of K a *simplex*. If we need to be specific, we add the dimension, calling a vertex a 0-simplex, an edge a 1-simplex, and a triangle a 2-simplex. A *face* of a simplex τ is a simplex $\sigma \subseteq \tau$. For example, a triangle has seven faces, its three vertices, its two edges, and itself. We can now state Condition I more succinctly: if σ is a face of τ and $\tau \in K$ then $\sigma \in K$. To talk about

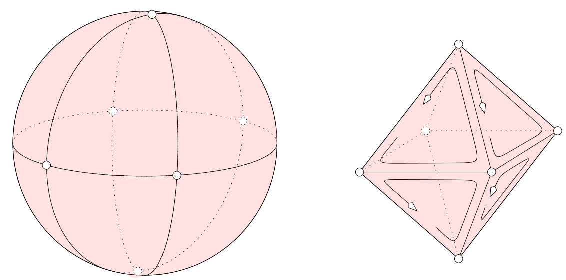

Figure 81: A triangulation of the sphere. The eight triangles are glued to form the boundary of an octahedron which is homeomorphic to the sphere.

the inverse of the face relation, we define the *star* of a simplex σ as the set of simplices that contain σ as a face, $\mathrm{St}\,\sigma = \{\tau \in K \mid \sigma \subseteq \tau\}$. Sometimes we think of the star as a set of simplices and sometimes as a set of points, namely the union of interiors of the simplices in the star. With the latter interpretation, we can now express Condition II more succinctly: the star of every simplex in K is homeomorphic to $\mathbb{R}^2$.

Data structure. When we store a 2-manifold, it is useful to keep track of which side we are facing and where we are going so that we can move around efficiently. The core piece of our data structure is a representation of the symmetry group of a triangle. This group is isomorphic to the group of permutations of three elements, the vertices of the triangle. We call each permutation an *ordered triangle* and use cyclic shifts and transpositions to move between them; see Figure 82. We store

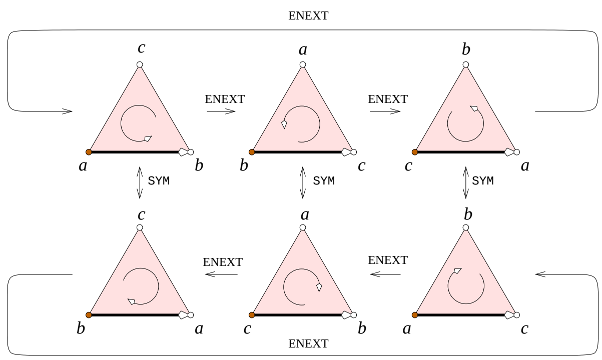

Figure 82: The symmetry group of the triangle consists of six ordered versions. Each ordered triangle has a lead vertex and a lead directed edge.

the entire symmetry group in a single node of an abstract graph, with arcs between neighboring triangles. Furthermore, we store the vertices in a linear array, $V[1..n]$. For each ordered triangle, we store the index of the lead vertex and a pointer to the neighboring triangle that shares the same directed lead edge. A pointer in this context is the address of a node together with a three-bit integer, ι, that identifies the ordered version of the triangle we refer to. Suppose for example that we identify the ordered versions $abc, bca, cab, bac, cba, acb$ of a triangle with $\iota = 0, 1, 2, 4, 5, 6$, in this sequence. Then we can move between different ordered versions of the same triangle using the following functions.

`ordTri` ENEXT(μ, ι)
  `if` $\iota \leq 2$ `then return` $(\mu, (\iota + 1) \bmod 3)$
    `else return` $(\mu, (\iota + 1) \bmod 3 + 4)$
  `endif`.

`ordTri` SYM(μ, ι)
  `return` $(\mu, (\iota + 4) \bmod 8)$.

To get the index of the lead vertex, we use the integer function ORG(μ, ι) and to get the pointer to the neighboring triangle, we use FNEXT(μ, ι).

Orientability. A 2-manifold is *orientable* if it has two distinct sides, that is, if we move around on one we stay there and never cross over to the other side. The one example of a non-orientable manifold we have seen so far is the Möbious strip in Figure 80. There are also non-orientable, compact 2-manifolds (without boundary), as we can see in Figure 83. We use the data structure to decide whether or

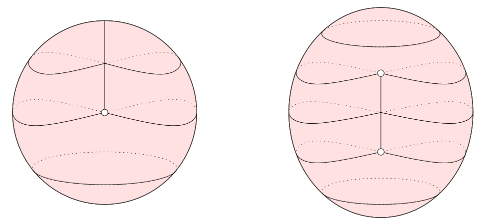

Figure 83: Two non-orientable, compact 2-manifolds, the projective plane on the left and the Klein bottle on the right.

not a 2-manifold is orientable. Note that the cyclic shift partitions the set of six ordered triangles into two *orientations*, each consisting of three triangles. We say two neighboring triangles are *consistently oriented* if they disagree on the direction of the shared edge, as in Figure 81. Using depth-first search, we visit all triangles and orient them consistently, if possible. At the first visit, we orient the triangle consistent with the preceding, neighboring triangle. At subsequence visits, we check for consistent orientation.

`boolean` ISORNTBL(μ, ι)
  `if` μ is unmarked `then`
    mark μ; choose the orientation that contains ι;
    $b_x =$ ISORNTBL(FNEXT(SYM(μ, ι)));
    $b_y =$ ISORNTBL(FNEXT(ENEXT(SYM(μ, ι))));
    $b_z =$ ISORNTBL(FNEXT(ENEXT^2(SYM(μ, ι))));
    `return` b_x `and` b_y `and` b_z
  `else`
    `return` [orientation of μ contains ι]
  `endif`.

There are two places where we return a boolean value. At the second place, it indicates whether or not we have consistent orientation in spite of the visited triangle being oriented prior to the visit. At the first place, the boolean value indicates whether or not we have found a contradiction to orientablity so far. A value of FALSE anywhere during the computation is propagated to the root of the search tree telling us that the 2-manifold is non-orientable. The running time is proportional to the number of triangles in the triangulation of the 2-manifold.

Classification. For the sphere and the torus, it is easy to see how to make them out of a sheet of paper. Twisting the paper gives a non-orientable 2-manifold. Perhaps most difficult to understand is the projective plane. It is obtained by gluing each point of the sphere to its antipodal point. This way, the entire northern hemisphere is glued to the southern hemisphere. This gives the disk except that we still need to glue points of the bounding circle (the equator) in pairs, as shown in the third paper construction in Figure 84. The Klein bottle is easier to imagine as it is obtained by twisting the paper just once, same as in the construction of the Möbius strip.

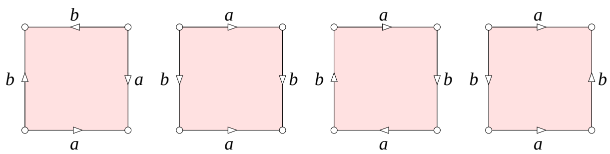

Figure 84: From left to right: the sphere, the torus, the projective plane, and the Klein bottle.

There is a general method here that can be used to classify the compact 2-manifolds. Given two of them, we construct a new one by removing an open disk each and gluing the 2-manifolds along the two circles. The result is called the *connected sum* of the two 2-manifolds, denoted as $\mathbb{M}\#\mathbb{N}$. For example, the double torus is the connected sum of two tori, $\mathbb{T}^2\#\mathbb{T}^2$. We can cut up the g-fold torus into a flat sheet of paper, and the canonical way of doing this gives a $4g$-gon with edges identified in pairs as shown in Figure 85 on the left. The number g is called the *genus* of the manifold. Similarly, we can get new non-orientable

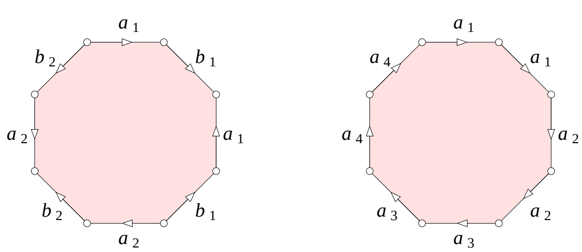

Figure 85: The polygonal schema in standard form for the double torus and the double Klein bottle.

manifolds from the projective plane, $\mathbb{P}^2$, by forming connected sums. Cutting up the g-fold projective plane gives a $2g$-gon with edges identified in pairs as shown in Figure 85 on the right. We note that the constructions of the projective plane and the Klein bottle in Figure 84 are both not in standard form. A remarkable result which is now more than a century old is that every compact 2-manifold can be cut up to give a standard polygonal schema. This implies a classification of the possibilities.

CLASSIFICATION THEOREM. The members of the families $\mathbb{S}^2, \mathbb{T}^2, \mathbb{T}^2\#\mathbb{T}^2, \ldots$ and $\mathbb{P}^2, \mathbb{P}^2\#\mathbb{P}^2, \ldots$ are non-homeomorphic and they exhaust the family of compact 2-manifolds.

Euler characteristic. Suppose we are given a triangulation, K, of a compact 2-manifold, $\mathbb{M}$. We already know how to decide whether or not $\mathbb{M}$ is orientable. To determine its type, we just need to find its genus, which we do by counting simplices. The *Euler characteristic* is

$$\chi \quad = \quad \#\text{vertices} - \#\text{edges} + \#\text{triangles}.$$

Let us look at the orientable case first. We have a $4g$-gon which we triangulate. This is a planar graph with $n - m + \ell = 2$. However, $2g$ edge are counted double, the $4g$ vertices of the $4g$-gon are all the same, and the outer face is not a triangle in K. Hence,

$$\begin{aligned}\chi &= (n - 4g + 1) - (m - 2g) + (\ell - 1) \\ &= (n - m + \ell) - 2g\end{aligned}$$

which is equal to $2 - 2g$. The same analysis can be used in the non-orientable case in which we get $\chi = (n - 2g + 1) - (m - g) + (\ell - 1) = 2 - g$. To decide whether two compact 2-manifolds are homeomorphic it suffices to determine whether they are both orientable or both non-orientable and, if they are, whether they have the same Euler characteristic. This can be done in time linear in the number of simplices in their triangulations.

This result is in sharp contrast to the higher-dimensional case. The classification of compact 3-manifolds has been a longstanding open problem in Mathematics. Perhaps the recent proof of the Poincaré conjecture by Perelman brings us close to a resolution. Beyond three dimensions, the situation is hopeless, that is, deciding whether or not two triangulated compact manifolds of dimension four or higher are homeomorphic is undecidable.