

22 Alpha Shapes

Many practical applications of geometry have to do with the intuitive but vague concept of the shape of a finite point set. To make this idea concrete, we use the distances between the points to identify subcomplexes of the Delaunay triangulation that represent that shape at different levels of resolution.

Union of disks. Let S be a set of n points in \mathbb{R}^2 . For each $r \geq 0$, we write $B_u(r) = \{x \in \mathbb{R}^2 \mid \|x - u\| \leq r\}$ for the closed disk with center u and radius r . Let $\mathbb{U}(r) = \bigcup_{u \in S} B_u(r)$ be the union of the n disks. We decompose this union into convex sets of the form $R_u(r) = B_u(r) \cap V_u$. Then

- (i) $R_u(r)$ is closed and convex for every point $u \in S$ and every radius $r \geq 0$;
- (ii) $R_u(r)$ and $R_v(r)$ have disjoint interiors whenever the two points, u and v , are different;
- (iii) $\mathbb{U}(r) = \bigcup_{u \in S} R_u(r)$.

We illustrate this decomposition in Figure 105. Each region $R_u(r)$ is the intersection of $n - 1$ closed half-planes and a closed disk. All these sets are closed and convex, which implies (i). The Voronoi regions have disjoint interiors, which implies (ii). Finally, take a point $x \in \mathbb{U}(r)$ and let u be a point in S with $x \in V_u$. Then $x \in B_u(r)$ and therefore $x \in R_u(r)$. This implies (iii).

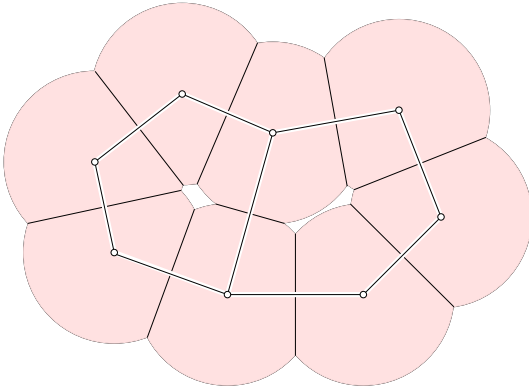


Figure 105: The Voronoi decomposition of a union of eight disks in the plane and superimposed dual alpha complex.

Nerve. Similar to defining the Delaunay triangulation as the dual of the Voronoi diagram, we define the alpha com-

plex as the dual of the Voronoi decomposition of the union of disks. This time around, we do this more formally. Letting C be a finite collection of sets, the *nerve* of C is the system of subcollections that have a non-empty common intersection,

$$\text{Nrv } C = \{X \subseteq C \mid \bigcap X \neq \emptyset\}.$$

This is an abstract simplicial complex since $\bigcap X \neq \emptyset$ and $Y \subseteq X$ implies $\bigcap Y \neq \emptyset$. For example, if C is the collection of Voronoi regions then $\text{Nrv } C$ is an abstract version of the Delaunay triangulation. More specifically, this is true provide the points are in general position and in particular no four points lie on a common circle. We will assume this for the remainder of this section. We say the Delaunay triangulation is a *geometric realization* of $\text{Nrv } C$, namely the one obtained by mapping each Voronoi region (a vertex in the abstract simplicial complex) to the generating point. All edges and triangles are just convex hulls of their incident vertices. To go from the Delaunay triangulation to the alpha complex, we substitute the regions $R_u(r)$ for the V_u . Specifically,

$$\text{Alpha}(r) = \text{Nrv } \{R_u(r) \mid u \in S\}.$$

Clearly, this is isomorphic to a subcomplex of the nerve of Voronoi regions. We can therefore draw $\text{Alpha}(r)$ as a subcomplex of the Delaunay triangulation; see Figure 105. We call this geometric realization of $\text{Alpha}(r)$ the *alpha complex* for radius r , denoted as $A(r)$. The *alpha shape* for the same radius is the underlying space of the alpha complex, $|A(r)|$.

The nerve preserves the way the union is connected. In particular, their Betti numbers are the same, that is, $\beta_p(\mathbb{U}(r)) = \beta_p(A(r))$ for all dimensions p and all radii r . This implies that the union and the alpha shape have the same number of components and the same number of holes. For example, in Figure 105 both have one component and two holes. We omit the proof of this property.

Filtration. We are interested in the sequence of alpha shapes as the radius grows from zero to infinity. Since growing r grows the regions $R_u(r)$, the nerve can only get bigger. In other words, $A(r) \subseteq A(s)$ whenever $r \leq s$. There are only finitely many subcomplexes of the Delaunay triangulation. Hence, we get a finite sequence of alpha complexes. Writing A_i for the i -th alpha complex, we get the following nested sequence,

$$S = A_1 \subset A_2 \subset \dots \subset A_k = D,$$

where D denotes the Delaunay triangulation of S . We call such a sequence of complexes a *filtration*. We illustrate this construction in Figure 106. The sequence of al-

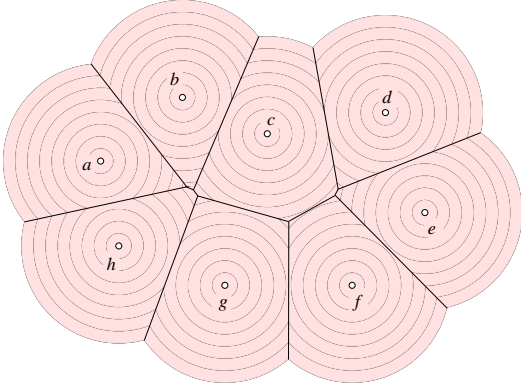


Figure 106: A finite sequence of unions of disks, all decomposed by the same Voronoi diagram.

pha complexes begins with a set of n isolated vertices, the points in S . To go from one complex to the next, we either add an edge, we add a triangle, or we add a pair consisting of a triangle with one of its edges. In Figure 106, we begin with eight vertices and get the following sequence of alpha complexes.

$$\begin{aligned} A_1 &= \{a, b, c, d, e, f, g, h\}; \\ A_2 &= A_1 \cup \{ah\}; \\ A_3 &= A_2 \cup \{bc\}; \\ A_4 &= A_3 \cup \{ab, ef\}; \\ A_5 &= A_4 \cup \{de\}; \\ A_6 &= A_5 \cup \{gh\}; \\ A_7 &= A_6 \cup \{cd\}; \\ A_8 &= A_7 \cup \{fg\}; \\ A_9 &= A_8 \cup \{cg\}. \end{aligned}$$

Going from A_7 to A_8 , we get for the first time a 1-cycle, which bounds a hole in the embedding. In A_9 , this hole is cut into two. This is the alpha complex depicted in Figure 105. We continue.

$$\begin{aligned} A_{10} &= A_9 \cup \{cf\}; \\ A_{11} &= A_{10} \cup \{abh, bh\}; \\ A_{12} &= A_{11} \cup \{cde, ce\}; \\ A_{13} &= A_{12} \cup \{cfg\}; \\ A_{14} &= A_{13} \cup \{cef\}; \\ A_{15} &= A_{14} \cup \{bch, ch\}; \\ A_{16} &= A_{15} \cup \{cgh\}. \end{aligned}$$

At this moment, we have a triangulated disk but not yet the entire Delaunay triangulation since the triangle bcd and the edge bd are still missing. Each step is generic except when we add two equally long edges to A_3 .

Compatible ordering of simplices. We can represent the entire filtration of alpha complexes compactly by sorting the simplices in the order they join the growing complex. An ordering $\sigma_1, \sigma_2, \dots, \sigma_m$ of the Delaunay simplices is *compatible* with the filtration if

1. the simplices in A_i precede the ones not in A_i for each i ;
2. the faces of a simplex precede the simplex.

For example, the sequence

$$\begin{aligned} &a, b, c, d, e, f, g, h; ah; bc; ab, ef; \\ &de; gh; cd; fg; cg; cf; bh, abh; ce, \\ &cde; cfg; cef; ch, bch; cgh; bd; bcd \end{aligned}$$

is compatible with the filtration in Figure 106. Every alpha complex is a prefix of the compatible sequence but not necessarily the other way round. Condition 2 guarantees that every prefix is a complex, whether an alpha complex or not. We thus get a finer filtration of complexes

$$\emptyset = K_0 \subset K_1 \subset \dots \subset K_m = D,$$

where K_i is the set of simplices from σ_1 to σ_i . To construct the compatible ordering, we just need to compute for each Delaunay simplex the radius $r_i = r(\sigma_i)$ such that $\sigma_i \in A(r)$ iff $r \geq r_i$. For a vertex, this radius is zero. For a triangle, this is the radius of the circumcircle. For

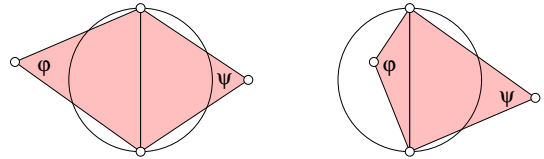


Figure 107: Left: the middle edge belongs to two acute triangles. Right: it belongs to an obtuse and an acute triangle.

an edge, we have two cases. Let φ and ψ be the angles opposite the edge σ_i inside the two incident triangles. We have $\varphi + \psi > 180^\circ$ because of the empty circle property.

CASE 1. $\varphi < 90^\circ$ and $\psi < 90^\circ$. Then $r_i = r(\sigma_i)$ is half the length of the edge.

CASE 2. $\varphi \geq 90^\circ$. Then $r_i = r_j$, where σ_j is the incident triangle with angle φ .

Both cases are illustrated in Figure 107. In Case 2, the edge σ_i enters the growing alpha complex together with the triangle σ_j . The total number of simplices in the Delaunay triangulation is $m < 6n$. The threshold radii can be computed in time $O(n)$. Sorting the simplices into the compatible ordering can therefore be done in time $O(n \log n)$.

Betti numbers. In two dimensions, Betti numbers can be computed directly, without resorting to boundary matrices. The only two possibly non-zero Betti numbers are β_0 , the number of components, and β_1 , the number of holes. We compute the Betti numbers of K_j by adding the simplices in order.

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 $\beta_0 = \beta_1 = 0;$ 
for  $i = 1$  to  $j$  do
  switch dim  $\sigma_i$ :
    case 0:  $\beta_0 = \beta_0 + 1;$ 
    case 1: let  $u, v$  be the endpoints of  $\sigma_i$ ;
      if FIND( $u$ ) = FIND( $v$ ) then  $\beta_1 = \beta_1 + 1$ 
      else  $\beta_0 = \beta_0 - 1;$ 
      UNION( $u, v$ )
    endif
    case 2:  $\beta_1 = \beta_1 - 1$ 
  endswitch
endfor.
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All we need is tell apart the two cases when σ_i is an edge. This is done using a union-find data structure maintaining the components of the alpha complex in amortized time $\alpha(n)$ per simplex. The total running time of the algorithm for computing Betti numbers is therefore $O(n\alpha(n))$.