1 Trees (10 points)

Show that in any $n$-node undirected tree, there is a single node (called a separator) whose removal partitions the nodes into two sets $A$ and $B$ such that each set has at most $\frac{2n}{3}$ nodes, and no edge connects a node in $A$ with a node in $B$. Hint: start at any node in the tree, and “walk” towards a better separator, if necessary.

2 Minimum Spanning Trees (10 points)

Show that if every edge in an undirected graph $G$ has a distinct weight then there is a unique minimum-weight spanning tree $T$ for $G$. Hint: study the lecture on minimum-weight spanning trees.

3 Graphs (10 points)

Show that if the maximum degree of any node in an undirected graph $G$ is $\delta$, then the nodes of $G$ can be colored using $\delta + 1$ colors so that no edge connects two nodes of the same color. Hint: you are thinking too hard if you need a hint on this one.

4 Prefix Sum (10 points)

In a segmented prefix sum, there is an input sequence $A = a_0, a_1, \cdots, a_{n-1}$ and a binary associative operator $\oplus$ over the elements in $A$, and a second sequence $B = b_0, b_1, \cdots, b_{n-1}$, where each element of $B$ is either 0 or 1. The second sequence, $B$, is used to segment $A$ into smaller subsequences such that an independent prefix sum is performed for each subsequence. In particular, if $b_i = 1$, then a new subsequence starts at $a_i$. As an example consider the sequences $A = 1, 3, 6, 2, 5, 3$ and $B = 1, 0, 0, 1, 0, 0$. If $\oplus$ is integer addition, then the output of the segmented prefix sum is $1, 4, 10, 2, 7, 10$. Show that segmented prefix sum is just a special case of prefix sum. Do this by showing how to map the inputs and operator of a segmented prefix sum to the inputs and operator of a standard prefix sum. Be sure to show that your operator is associative.
5   Graphs (10 points)
A graph with vertices numbers 1 through \( n \) can be represented using two sequences \( A \) and \( B \) as follows. The first sequence, \( A \), lists the neighbors of each vertex, one after another. For example, for the complete graph on three vertices, \( A \) is 2, 3, 1, 3, 1, 2. The second sequence segments the first so that there is a distinct subsequence (list of neighbors) for each vertex. For the complete graph on three vertices, \( B \) is 1, 0, 1, 0, 1, 0. Show how to compute, for each vertex, the largest numbered neighbor. Your algorithm may perform a constant number of segmented (or standard) prefix sums, and a constant (not linear) number of additional operations.

6   Long Division (10 points)
Show that every rational number has a repeated decimal representation, i.e., a representation of the form \( 1.4569 = 1.456969696969 \cdots \). Hint: analyze the standard long-division algorithm and use the pigeonhole principle

7   Group Theory (10 points)
Prove that the set of rational numbers is closed under addition and multiplication.

8   Bipartite Graphs (5 points)
Suppose that there are five young men and five young women on an island. Each man is willing to marry some of the women on the island and each woman is willing to marry any man who is willing to marry her. Suppose that Sandeep is willing to marry Tina and Vandana; Barry is willing to marry Tina, Xia, and Uma; Teja is willing to marry Tina and Zelda; Anil is willing to marry Vandana and Zelda; and Emilio is willing to marry Tina and Zelda. Use Marriage Theorem to show there is no matching of the young men and young women on the island such that each young man is matched with a young woman he is willing to marry.

9   Minimum Spanning Trees (5 points)
Show that an edge with smallest weight in a connected weighted graph must be part of any minimum spanning tree.

10  Planar Graphs (5 points)
Suppose that a connected bipartite planar simple graph has \( e \) edges and \( v \) vertices. Show that \( e \leq 2v - 4 \) if \( v \geq 3 \)