- bipartite graph

- A bipartite graph $G = (V_1, V_2, E)$, $E$ is a subset of $(i,j) \in V_1 \times V_2$.

  $(V_1: \text{courses} \quad V_2: \text{classrooms} \quad (i,j) \in E: \text{course } i \text{ can be assigned to classroom } j)$

- A matching $M$ is a subset of $E$, such that edges in $M$ do not share vertices.

  $M: \{(1,1), (2,3), (3,4)\}$

  $\vert M \vert = 3$

  blue: augmenting path

The size of a matching $M$ is just the # of edges in $M$.

- Given a bipartite graph $G$ and matching $M$.

  - an edge $e$ is matched if $e \in M$

    unmatched if $e \notin M$

  - a vertex is matched if it's connected to some $e \in M$

    unmatched otherwise.

  - augmenting path $P$ is a path from an unmatched vertex in $V_1$, to an unmatched vertex on $V_2$, and the edges alternate between unmatched and matched.

Claim: An augmenting path $P$ has an odd # of edges, and it has exactly 1 more unmatched edges than matched edges.

- XOR operation: If $A, B$ are two subsets of edges,
A \oplus B is also a subset of edges
\[ e \in A \oplus B \text{ if } \begin{cases} \exists e \in A, e \notin B \\ \exists e \in B, e \notin A \end{cases} \]

Claim: if \( P \) is an augmenting path for \( M \), then \( M' = M \oplus P \) is also a matching, and \( |M'| = |M| + 1 \)

- Example of DFS for augmenting path

- Then for correctness:

Proof by contradiction:
assume there is a larger matching \( M^* \) (\( |M^*| > |M| \))
\[ \Rightarrow \text{there is an augmenting path}. \]

Proof: Look at \( M \oplus M^* \)
\[ M \oplus M^* : \begin{cases} \text{each vertex is connected to } \leq 2 \text{ edges} \\ \text{union of vertex-disjoint paths and cycles} \end{cases} \]
- in a cycle: same # of edges in $M, M^*$
  - in a path: either $M^*$ has 1 more edge or $M$ has 1 more edge

Case 2 must happen because $|M^*| > |M|