- Dynamic array

\[
\begin{align*}
\text{initial} & \quad \text{length} = 0 \quad \text{capacity} = 1 \\
\text{add(1)} & \quad \begin{array}{c} 1 \end{array} \quad 1 \\
\text{add(2)} & \quad \begin{array}{c} 1 \ 2 \end{array} \quad 2 \quad 2 \ (2 \times 1) \\
\text{add(3)} & \quad \begin{array}{c} 1 \ 2 \ 3 \end{array} \quad 3 \quad 4 \ (2 \times 2) \\
\text{add(4)} & \quad \begin{array}{c} 1 \ 2 \ 3 \ 4 \end{array} \quad 4 \quad 4 \\
\text{add(5)} & \quad \begin{array}{c} 1 \ 2 \ 3 \ 4 \ 5 \ | \ | \ | \ | \ |
\end{array} \quad 5 \quad 8 \ (2 \times 4)
\end{align*}
\]

- Analyzing running time

- Time of add operation

- \((2^k+1)\)th add operation

\[
\text{running time } 2^{k+1}
\]

(allocation an array of size \(2^{k+1}\))

(copy first \(2^k\) elements)

(add the \(2^{k+1}\)th element)

- All other add operations

\[
\text{running time } 1 \quad \text{(change length, put the element into an empty slot)}
\]

- Aggregate method

Let \(t_i\) be running time of the \(i\)-th add operation

\[
\text{total running time } = \sum_{i=1}^{n} t_i
\]

\[
= \sum_{i=1}^{k+1} t_i + \sum_{i=k+1}^{\infty} 2^{k+1}
\]

(light operation)

\[
\leftarrow \sum_{i=1}^{k} t_i + \sum_{i=k+1}^{\infty} 2^{k+1}
\]

(heavy operation)
\[
\sum_{i=1}^{2^k+1} 1 + \sum_{k=1}^{l+1} 2^k \leq n + (2+4+8+\ldots+2^{l+1})
\]

Let \( l \) be the largest number such that \( 2^{l+1} \leq n \).

\[
\Rightarrow 2^{l+1} \leq 2n
\]

\[
\Rightarrow n + 2^{l+2} \leq 2n
\]

Amortized time \( = \frac{\text{total time}}{n} = 5 \Rightarrow O(1) \)

- Charging argument

between two heavy operations

\[
2^k + 1 \rightarrow 2^{k+1}
\]

we have \( 2^{k+1} - (2^k+1)-1 \) light operations

\[
= 2^{k-1}
\]

Cost for the \( 2^{k+1} \) operation is \( 2^{k+2} \)

\[
\frac{2^{k+2}}{2^{k-1}} = 4
\]

Save 4 units of time for each light operation when \( (2^{k+1}) \)th operation (heavy) happens.

I have saved \( (2^{k-1}) \cdot 4 = 2^{k+2} - 4 \) units of time.

I need to pay \( 2^{k+2} \).

So the additional money (time) to pay is just 4.

In summary:

\[
\begin{align*}
\text{For light operation: pay 1, save 4 (1+4=5) } \\
\text{Heavy: use all saving, pay 4 (4) }
\end{align*}
\]

Amortized cost \( \leq 5 = O(1) \)