CMSC 858S: Randomized Algorithms  
Fall 2001  
Handout 6: The vertex-connectivity of random graphs

In this short handout, we complete the discussion from class on the vertex-connectivity of random graphs. As mentioned in class, stronger results are known; see [1]. The discussion here is adapted from [2].

Recall the context. We choose a random graph $G$ from $G(n,p)$, where $p = p(n) \geq (\ln^2 n)/n$, say. Fix any constant $\delta > 0$, and let $k = \lceil (n-1)p(1-2\delta) \rceil$. We wish to prove that as $n \to \infty$, the probability that $G$ is not $k$-vertex-connected, tends to 0. We now zoom to the point at which we stopped this argument in class. Let $t = \lceil (n-1)p\delta \rceil$, and $s = k - 1$. We aim to show that

$$q = \binom{n}{s} \cdot \sum_{i=t}^{[n-s]/2} \binom{n-s}{i} \cdot (1-p)^{i[n-s-i]}$$

tends to 0 as $n \to \infty$, under the above-seen assumptions on $p$, $\delta$ etc.

We start with some simplifications. We have

$$q = \binom{n}{\min\{s, n-s\}} \cdot \sum_{i=t}^{[n-s]/2} \binom{n-s}{i} \cdot (1-p)^{i[n-s-i]}$$

$$\leq n^{\min\{s, n-s\}} \cdot \sum_{i=t}^{[n-s]/2} \binom{n-s}{i} \cdot (1-p)^{i[n-s-i]}$$

$$\leq n^{\min\{s, n-s\}} \cdot \sum_{i=t}^{[n-s]/2} (n-s)^i \cdot e^{-ip(n-s-i)}$$

$$\leq n^{\min\{s, n-s\}} \cdot \sum_{i=t}^{[n-s]/2} (n-s)^i \cdot e^{-ip(n-s)/2} \quad \text{(since } i \leq (n-s)/2)$$

$$\leq n^{\min\{s, n-s\}} \cdot \sum_{i=t}^{[n-s]/2} (n-s)^i \cdot e^{-ip(n-s)/2}$$

$$= n^{\min\{s, n-s\}} \cdot \sum_{i=t}^{[n-s]/2} e^{i\ln(n-s) - p(n-s)/2}.$$  (1)

We now consider two cases. First suppose $s \leq n/2$. Since $n-s \geq n/2$, we get from (1), if $n$ is large enough, that

$$q \leq n^s \cdot \sum_{i=t}^{[n-s]/2} e^{i\ln(n-np)/4}$$

$$\leq n^s \cdot \sum_{i=t}^{[n-s]/2} e^{-\Omega(np)} \quad \text{(since } np \gg \ln n)$$

$$\leq e^{O(np\ln n - \Omega(np\delta))}$$

$$\to 0,$$

since $t \gg \ln n$ for any fixed $\delta > 0$, as $n \to \infty$.  

1
Next suppose \( s > n/2 \). Here, we must have \( p \geq 1/2 \); so, (1) shows that

\[
q \leq n^{n-s} \cdot \sum_{i=t}^{(n-s)/2} e^{i((n-s)-(n-s)/4)}.
\]

Note that \( n - s \geq 2\delta n \). So, since \( \delta \) is some positive constant, we have when \( n \) gets arbitrarily large that \( \ln(n - s) \ll (n - s)/4 \). Thus,

\[
q \leq n^{n-s} \cdot \sum_{i=t}^{(n-s)/2} e^{-\Omega(i|n-s|)} \\
\leq e^{O((n-s) \ln n - \Omega(t(n-s))} \\
\rightarrow 0,
\]

again since \( t \gg \ln n \) and since \( n - s \) is arbitrarily large as \( n \to \infty \).

References
