Date midterm.

**Treaps**

Review:
- Dictionaries for **ordered** sets
- Binary tree.
- Tree balancing by rotations
- drawbacks in geometry: rebuild on rotation

Returning to average case:
- Assign random “arrival orders” to keys
- Build tree **as if** arrived in that order
- Average case applies
- No rotations on searches

Choosing priorities
- define arrival by random priorities
- assume continuous distribution, fix.
- eg, use $2 \log n$ bits, w.h.p. no collisions

Treaps.
- tree has keys in heap order of priorities
- unique tree given priorities—follows from insertion order
- implement insert/delete etc.
- rotations to maintain heap property

Depth $d(x)$ analysis
- Tree is trace of a quicksort
- We proved $O(\log n)$ w.h.p.
- for $x$ rank $k$, $E[d(x)] = H_k + H_{n-k+1} - 1$
- $S^- = \{y \in S \mid y \leq x\}$
- $Q_x = $ ancestors of $x$
• Show $E[Q^-_x] = H_k$.

• to show: $y \in Q^-_x$ iff inserted before all $z$, $y < z \leq x$.

• deduce: item $j$ away has prob $1/j$. Add.

• Suppose $y \in Q^-_x$.
  – The inserted before $x$
  – Suppose some $z$ between inserted before $y$
  – Then $y$ in left subtree of $z$, $x$ in right, so not ancestor
  – Thus, $y$ before every $z$

• Suppose $y$ first
  – then $x$ follows $y$ on all comparisons (no $z$ splits
  – So ends up in subtree of $y$

Rotation analysis

• Insert/Delete time
  – define spines
  – equal left spine of right sub plus right spine of left sub
  – proof: when rotate up, on spine increments, other stays fixed.

• $R_x$ length of right spine of left subtree

• $E[R_x] = 1 - 1/k$ if rank $k$

• To show: $y \in R_x$ iff
  – inserted after $x$
  – all $z$, $y < z < x$, arrive after $y$.
  – if $z$ before $y$, then $y$ goes left, so not on spine

• deduce: if $r$ elts between, $r!$ of $(r + 2)!$ permutations work.

• So probability $1/r^2$.

• Expectation $\sum 1/(1 \cdot 2) + 1/(2 \cdot 3) + \cdots = 1 - 1/k$

• subtle: do analysis only on elements inserted in real-time before $x$, but now assume they arrive in random order in virtual priorities.
skip lists

- ruler intuition
- achieve with geometric variables
- backwards analysis of search path
- insert/delete time

### Shortest Paths

classical shortest paths.

- dijkstra’s algorithm
- floyd’s algorithm. similarity to matrix multiplication

### Matrices

- length 2 paths by squaring
- matrix multiplication. strassen.
- shortest paths by “funny multiplication.”
  - huge integer implementation
  - base-\((n + 1)\) integers

### Boolean matrix multiplication

- easy.
- gives objects at distance 2.
- gives n-mul algorithm for problem
- what about recursive?
- well can get to within 2: let \(T_k\) be boolean “distance less than or equal to \(2^k\). Squaring gives \(T_{k+1}\).
- what about exact?

### Seidel’s distance algorithm.

- log-size integers:
  - parities suffice:
    * square \(G\) to get adjacency \(A'\), distance \(D'\)
    - if \(D_{ij}\) even then \(D_{ij} = 2D'_{ij}\)
\[ \text{if } D_{ij} \text{ odd then } D_{ij} = 2D'_{ij} - 1 \]

- For neighbors \( i, k \),
  * \( D_{ij} - 1 \leq D_{kj} \leq D_{ij} + 1 \)
  * exists \( k \), \( D_{kj} = D_{ij} - 1 \)

- Parities
  * If \( D_{ij} \) even, then \( D'_{kj} \geq D'_{ij} \) for every neighbor \( k \)
  * If \( D_{ij} \) odd, then \( D'_{kj} \leq D'_{ij} \) for every neighbor \( k \), and strict for at least one
- Add
  * \( D_{ij} \) even iff \( S_{ij} = \sum_k D'_{kj} \geq D_{ij}d(i) \)
  * \( D_{ij} \) odd iff \( \sum_k D'_{kj} < D_{ij}d(i) \)
  * How determine? find \( S = AD' \)

To find paths: Witness product.

- easy case: unique witness
  - multiply column \( c \) by \( c \).
  - read off witness identity

- reduction to easy case:
  - suppose \( r \) columns have witness, where \( 2^k \leq r \leq 2^{k+1} \)
  - choose each column with probability \( 2^{-k} \).
  - prob. exactly one witness: \( r \cdot 2^{-k}(1 - 2^{-k})^{r-1} \geq (1/2)(1/e^2) \)

Mod 3:

- Recall some neighbor distance down by one
- so compute distances mod 3.
- suppose \( D_{ij} = 1 \) mod 3
- then look for \( k \) neighbor of \( i \) such that \( D_{kj} = 0 \) mod 3
- let \( D^{(s)}_{ij} = 1 \) iff \( D_{ij} = s \) mod 3
- than \( AD^{(s)} \) has \( ij = 1 \) iff a neighbor \( k \) of \( i \) has \( D^{(s)}_{kj} \)
- so, witness matrix mul!