Example

$L = \{a^n b a^n \mid n > 0\}$

Closure Properties

A set is closed over an operation if

$L_1, L_2 \in \text{class}$
$L_1 \text{ op } L_2 = L_3$
$\Rightarrow L_3 \in \text{class}$

Example

$L_1 = \{x \mid x \text{ is a positive even integer}\}$

L is closed under

- addition?
- multiplication?
- subtraction?
- division?

Closure of Regular Languages

Theorem 4.1 If $L_1$ and $L_2$ are regular languages, then

$L_1 \cup L_2$
$L_1 \cap L_2$
$L_1 L_2$
$L_1^*$

are regular languages.
Proof (sketch)

$L_1$ and $L_2$ are regular languages
$\Rightarrow \exists$ reg. expr. $r_1$ and $r_2$ s.t.
$L_1 = L(r_1)$ and $L_2 = L(r_2)$
$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
$\Rightarrow$ closed under union
$r_1 r_2$ is r.e. denoting $L_1 L_2$
$\Rightarrow$ closed under concatenation
$r_1^*$ is r.e. denoting $L_1$
$\Rightarrow$ closed under star-closure

complementation:
$L_1$ is reg. lang.
$\Rightarrow \exists$ DFA $M$ s.t. $L_1 = L(M)$
Construct $M'$ s.t.
final states in $M$ are nonfinal states in $M'$
nonfinal states in $M$ are final states in $M'$
$\Rightarrow$ closed under complementation

intersection:
$L_1$ and $L_2$ are reg. lang.
$\Rightarrow \exists$ DFA $M_1$ and $M_2$ s.t.
$L_1 = L(M_1)$ and $L_2 = L(M_2)$
$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$
$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$
Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$
$Q' = Q \times P$
$\delta'$:
$\delta'((q_i, p_j), a) = (q_k, p_l)$ if

$w \in L(M') \iff w \in L_1 \cap L_2$
$\Rightarrow$ closed under intersection
Regular languages are closed under

- reversal \( L^R \)
- difference \( L_1 - L_2 \)
- right quotient \( L_1 / L_2 \)
- homomorphism \( h(L) \)

**Right quotient**

Def: \( L_1 / L_2 = \{ x | xy \in L_1 \text{ for some } y \in L_2 \} \)

Example:

\[
\begin{align*}
L_1 &= \{ a^*b^* \cup b^*a^* \} \\
L_2 &= \{ b^n | n \text{ is even, } n > 0 \} \\
L_1 / L_2 &=
\end{align*}
\]

**Theorem** If \( L_1 \) and \( L_2 \) are regular, then \( L_1 / L_2 \) is regular.

**Proof** (sketch)

\( \exists \) DFA \( M = (Q, \Sigma, \delta, q_0, F) \) s.t. \( L_1 = L(M) \).

Construct DFA \( M' = (Q, \Sigma, \delta, q_0, F') \)

For each state \( i \) do

- Make \( i \) the start state (representing \( L_i' \))
- if \( L_i' \cap L_2 \neq \emptyset \) then
  - put \( q_i \) in \( F' \) in \( M' \)

QED.
Homomorphism

Def. Let \( \Sigma, \Gamma \) be alphabets. A homomorphism is a function

\[
h: \Sigma \rightarrow \Gamma^*
\]

Example:

\[
\Sigma = \{a, b, c\}, \Gamma = \{0, 1\},
\]

\[
\begin{align*}
h(a) &= 11 \\
h(b) &= 00 \\
h(c) &= 0
\end{align*}
\]

\[
h(bc) =
\]

\[
h(ab^*) =
\]

Questions about regular languages:

L is a regular language.

- Given \( L, \Sigma, w \in \Sigma^* \), is \( w \in L \)?

- Is \( L \) empty?

- Is \( L \) infinite?

- Does \( L_1 = L_2 \)?
Identifying Nonregular Languages

If a language L is finite, is L regular?

If L is infinite, is L regular?

- \( L_1 = \{a^nb^m | n > 0, m > 0\} = aa^*bb^* \)
- \( L_2 = \{a^n b^n | n > 0\} \)

Prove that \( L_2 = \{a^n b^n | n > 0\} \) is ?

- Proof:
Pumping Lemma: Let $L$ be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

- $|xy| \leq m$
- $|y| \geq 1$
- $xy^iz \in L$ for all $i \geq 0$

Meaning: Every long string in $L$ (the constant $m$ above corresponds to the finite number of states in $M$ in the previous proof) can be partitioned into three parts such that the middle part can be “pumped” resulting in strings that must be in $L$.

To Use the Pumping Lemma to prove $L$ is not regular:

- Proof by Contradiction.
  Assume $L$ is regular.
  $\Rightarrow$ $L$ satisfies the pumping lemma.
  Choose a long string $w$ in $L$, $|w| \geq m$. (The choice of the string is crucial. Must pick a string that will yield a contradiction).
  Show that there is NO division of $w$ into $xyz$ (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $xy^iz \in L \forall i \geq 0$.
  The pumping lemma does not hold. Contradiction!
  $\Rightarrow$ $L$ is not regular. QED.

Example $L=\{a^nb^n|n>0\}$

$L$ is not regular.

- Proof:
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w =$ where $m$ is the constant in the pumping lemma. (Note that $w$ must be chosen such that $|w| \geq m$.)
  The only way to partition $w$ into three parts, $w = xyz$, is such that $x$ contains 0 or more $a$’s, $y$ contains 1 or more $a$’s, and $z$ contains 0 or more $a$’s concatenated with $cb^m$. This is because of the restrictions $|xy| \leq m$ and $|y| > 0$. So the partition is:

  It should be true that $xy^iz \in L$ for all $i \geq 0$. 
Example \( L = \{a^nb^{n+s}c^s | n, s > 0\} \)

\( L \) is not regular.

- **Proof:**
  Assume \( L \) is regular.
  \( \Rightarrow \) the pumping lemma holds.
  Choose \( w = \)
  
  The only way to partition \( w \) into three parts, \( w = xyz \), is such that \( x \) contains 0 or more \( a \)'s, \( y \) contains 1 or more \( a \)'s, and \( z \) contains 0 or more \( a \)'s concatenated with the rest of the string \( b^{m+s}c^s \).
  This is because of the restrictions \( |xy| \leq m \) and \( |y| > 0 \). So the partition is:

---

Example \( \Sigma = \{a, b\}, L = \{w \in \Sigma^* \ | \ n_a(w) > n_b(w)\} \)

\( L \) is not regular.

- **Proof:**
  Assume \( L \) is regular.
  \( \Rightarrow \) the pumping lemma holds.
  Choose \( w = \)

  So the partition is:
Example \( L = \{a^3b^n c^{n-3} \mid n > 3\} \)

L is not regular.

- Proof:
  Assume \( L \) is regular. \( \Rightarrow \) the pumping lemma holds.
  Choose \( w = a^3b^m c^{m-3} \) where \( m \) is the constant in the pumping lemma. There are three ways to partition \( w \) into three parts, \( w = xyz \). 1) \( y \) contains only \( a \)'s 2) \( y \) contains only \( b \)'s and 3) \( y \) contains \( a \)'s and \( b \)'s

  We must show that each of these possible partitions lead to a contradiction. (Then, there would be no way to divide \( w \) into three parts s.t. the pumping lemma constraints were true).

  **Case 1:** (\( y \) contains only \( a \)'s). Then \( x \) contains 0 to 2 \( a \)'s, \( y \) contains 1 to 3 \( a \)'s, and \( z \) contains 0 to 2 \( a \)'s concatenated with the rest of the string \( b^m c^{m-3} \), such that there are exactly 3 \( a \)'s. So the partition is:

  \[
  x = a^k \quad y = a^j \quad z = a^{3-k-j} b^m c^{m-3}
  \]

  where \( k \geq 0, j > 0, \text{ and } k + j \leq 3 \) for some constants \( k \) and \( j \).

  It should be true that \( xy^iz \in L \) for all \( i \geq 0 \).

  \[
  xy^2z = (x)(y^i)(z) = (a^k)(a^j)(a^{3-j-k} b^m c^{m-3}) = a^{3+j} b^m c^{m-3} \not\in L \text{ since } j > 0, \text{ there are too many } a \text{'s. \( \Rightarrow \) contradiction!}
  \]

  **Case 2:** (\( y \) contains only \( b \)'s) Then \( x \) contains 3 \( a \)'s followed by 0 or more \( b \)'s, \( y \) contains 1 to \( m - 3 \) \( b \)'s, and \( z \) contains 3 to \( m - 3 \) \( b \)'s concatenated with the rest of the string \( c^{m-3} \). So the partition is:

  \[
  x = a^3 b^k \quad y = b^j \quad z = b^{m-k-j} c^{m-3}
  \]

  where \( k \geq 0, j > 0, \text{ and } k + j \leq m - 3 \) for some constants \( k \) and \( j \).

  It should be true that \( xy^iz \in L \) for all \( i \geq 0 \).

  \[
  xy^2z = a^3 b^j a^j \in L \text{ since } j > 0, \text{ there are too few } b \text{'s. \( \Rightarrow \) contradiction!}
  \]

  **Case 3:** (\( y \) contains \( a \)'s and \( b \)'s) Then \( x \) contains 0 to 2 \( a \)'s, \( y \) contains 1 to 3 \( a \)'s, and 1 to \( m - 3 \) \( b \)'s, \( z \) contains 3 to \( m - 1 \) \( b \)'s concatenated with the rest of the string \( c^{m-3} \). So the partition is:

  \[
  x = a^{3-k} \quad y = a^k b^j \quad z = b^{m-j} c^{m-3}
  \]

  where \( 3 \geq k > 0, \text{ and } m - 3 \geq j > 0 \) for some constants \( k \) and \( j \).

  It should be true that \( xy^iz \in L \) for all \( i \geq 0 \).

  \[
  xy^2z = a^3 b^j a^k b^m c^{m-3} \not\in L \text{ since } j, k > 0, \text{ there are } b \text{'s before } a \text{'s. \( \Rightarrow \) There is no partition of } w.
  \]

  \( \Rightarrow \) \( L \) is not regular!. QED.
To Use Closure Properties to prove L is not regular:

Using closure properties of regular languages, construct a language that should be regular, but for which you have already shown is not regular. Contradiction!

- **Proof Outline:**
  Assume L is regular.
  Apply closure properties to L and other regular languages, constructing L’ that you know is not regular.
  closure properties ⇒ L’ is regular.
  Contradiction!
  L is not regular. QED.

**Example** $L = \{a^{3i}b^n c^{n-3} | n > 3\}$

L is not regular.

- **Proof:** (proof by contradiction)
  Assume L is regular.
  Define a homomorphism $h : \Sigma \rightarrow \Sigma^*$
  $h(a) = a \quad h(b) = a \quad h(c) = b$
  $h(L) = \ldots$
Example \( L = \{a^n b^m a^m \mid m \geq 0, n \geq 0\} \)

\( L \) is not regular.

- **Proof:** (proof by contradiction)
  
  Assume \( L \) is regular.

---

Example: \( L_1 = \{a^n b^n a^n \mid n > 0\} \)

\( L_1 \) is not regular.

- **Proof:**
  
  Assume \( L_1 \) is regular.
  
  Goal is to try to construct \( \{a^n b^n \mid n > 0\} \) which we know is not regular.
  
  Let \( L_2 = \{a^*\} \). \( L_2 \) is regular.
  
  By closure under right quotient, \( L_3 = L_1 \setminus L_2 = \{a^n b^n a^p \mid 0 \leq p \leq n, n > 0\} \) is regular.
  
  By closure under intersection, \( L_4 = L_3 \cap \{a^* b^*\} = \{a^n b^n \mid n > 0\} \) is regular.
  
  Contradiction, already proved \( L_4 \) is not regular!
  
  Thus, \( L_1 \) is not regular. QED.