There are 50 true/false questions. Mark each question T, F and give a brief reason in the space below. Each person must work individually and the honor code applies.

[1] In HW 1, the system of nonlinear equations, \( g(u) \), was solved by nonlinear Gauss-Seidel, which required computing the Jacobian, \( g'(u) \).

[2] Any rational number \( r = a/b \), \( a \) and \( b \neq 0 \) integer, can be expressed exactly in fixed finite precision.

[3] Suppose \( f(x) \) is Lipschitz continuous on \([a,b]\) (i.e., \( |f(x)-f(y)| \leq L|x-y| \), \( x \) and \( y \) in \([a,b]\)) and \( f(a) \cdot f(b) < 0 \). Then there is a root \( r \) in \([a,b]\) with \( f(r) = 0 \).

[4] \( \sqrt{2} \neq a/b \), \( a \) and \( b \) integers.
[5] Let A be an $8 \times 7$ matrix in Matlab. Then $A(:,5)$ is the 5th column of A.

[6] Let A be an $m \times n$ real matrix. Then the columns of A are independent if $Ax = 0$ implies $x = 0$.

[7] Any nonsingular $n \times n$ matrix, A, can be factored as $A = LU$ where L is unit lower triangular and U is upper triangular with nonzero diagonal entries.

[8] A triangular matrix is nonsingular if and only if it has all diagonal entries nonzero.

[9] An LU factorization of an $n \times n$ matrix takes $\frac{1}{3} n^3 + O(n^2)$ arithmetic operations.

[10] An $n \times n$ symmetric positive definite matrix is nonsingular.
[11] An m x n matrix with m > n can have at most n independent columns.

[12] The secant-bisection method for solving a single equation, \( f(x) = 0 \), has a better convergence rate (per function evaluation) than Newton’s method if \( f \) and \( f' \) are equally expensive to evaluate.

[13] The columns of an n x n matrix of real numbers form a basis of the vector space \( \mathbb{R}^n \).

[14] The QR factorization of an m x n matrix, A, where the columns of Q are orthogonal and R is unit upper triangular is an algebraic statement of Gram-Schmidt orthogonalization.

Let \( V \) be an inner product space and \( W \) be a subspace. Let \( x \in V \) and \( w \) be the projection of \( x \) onto \( W \). Then:

[15] \( w \) is the best approximation to \( x \) in \( W \);

[16] \( x^T V x = w^T W w \);
[17] $y = x - w$ is orthogonal to any $z \in W$.

[18] $(y, 3w) > 0$.


[20] The Gauss-Seidel method for solving $Ax = b$ converges for any $x_0$ if $A$ is symmetric positive definite.

[21] If $\| \cdot \|$ is a norm on a vector space $V$ with $x$ and $y$ in $V$, then $\| x + y \| \leq \| x \| + \| y \|$.

[22] $1/10$ can be represented exactly in 64 bit binary floating point arithmetic.
[23] If \( A \) is \( m \times n \), \( m > n \), and has full rank, then \( A^T A \) has is nonsingular and 
\( A^T A x = A^T b \) are the normal equations for the least squares solution which 
minimizes \( \| Ax - b \| \).

The normal equations in [23] can be solved using:

[24] the shooting method by guessing values for \( x \) and then shooting forward to get \( b \);

[25] the singular value decomposition of \( b \).

[26] A splitting of \( A = M - N \) leads to a convergent iterative method, 
\( Mx_{n+1} = Nx_n + b \),
for solving \( Ax = b \) if \( \rho (M^{-1}N) < 1 \).

[27] The minimum degree algorithm is a method for solving systems of nonlinear 
equations.

[29] An orthogonal matrix U has $U^T = U^{-1}$.

[30] If A is a sparse n x n matrix, the graph of A represents the zero/nonzero structure of A.

[31] If the residual $r = Ax - b$ is small when trying to solve $Ax = b$, then the error is also small.

[32] An n x n matrix is similar (i.e., similarity transformation) to a diagonal matrix.

[33] The point of HW 3 was that the damped Newton’s method always converges to a zero of a system of nonlinear equations if the right damping strategy is used.

[34] The equation $\sin(x) + \sqrt{2} = 0$ has a root in $[0, 2\pi]$ and can be computed by the bisection method.
Any 1-step method for numerically solving an ODE-IVP is convergent as the (time) step-size decreases (ignoring roundoff).

The Vandermonde matrix arises in the polynomial interpolation problem.

The inequality

$$\| g_{k+1} \| \leq \| g_k \| \left\{ \|1 - t_k\| + \alpha_k t_k + \gamma_k t_k^2 \right\}$$

where

$$\alpha_k = \| g'_k x_k + g_k \| / \| g_k \|, \quad \gamma_k = \left( k_i^2 k_2 / 2 \right) \| g_k \|,$$

is important in developing the approximate Newton method and analysis. This is the context of questions [37] – [42].

The inequality is used to show we can make the norm decrease under certain conditions related to $\alpha_k, k_1$ and $k_2$.

$\alpha_k > 1$ is necessary for convergence.

We assumed that $\| x_k \| \leq k_1 \| g_k \|$. 

[40] \( k_2 \) is defined by the SVD.

[41] In the notation above, \( g'(u) \) is singular at a nonzero local minimum (\( g(u) \neq 0 \)).

[42] If an iterative method is used to solve the Newton equations, for example, Gauss-Seidel, the number of inner Gauss-Seidel iterations must approximately double to obtain asymptotic (as \( k \to \infty \)) quadratic convergence.

[42] Milne’s device is sometimes used to estimate the condition number of an \( n \times n \) matrix \( A \) in solving \( Ax = b \).

[43] TR-BDF is a method for solving a stiff system ODEs.

[44] In HW 4, the solution to the tridiagonal equations say, \( Tx = b \), was to be obtained using the matlab command \( x = \text{inv}(T) \ast b \).
In HW 4, the constant-j method led to Newton equations, $Tx = b$, where $T$ was triangular.

In HW 4, the Newton differential equation was solved by matlab's ode23, which allowed variable timesteps.

In HW 4, the third order RK method was used to provide a local truncation error estimator for the second order RK method.

Forward Euler is L-stable.

If $A$ is $m \times n$ with only $k<n$ independent columns, one can use the SVD to computer the pseudo-inverse $A^+$, and $x = A^+b$ will produce the best approximation to $\|Ax-b\|$ which minimizes $\|x\|_2$.

If $A$ is a sparse symmetric positive definite matrix, the graph $G(A)$ can be used to analyze the fill-in produced by sparse $LDL^T$ factorization.