HW 4: Solving Nonlinear Systems by Integrating Newton Differential Equation; BVP By Constant-j Discretization

Due: April 18, 2003

1. Overview

For this project each team will submit one detailed report, but each team member must know what is “going on.” Each team will present their work in a 30 minute session on April 18, the due date. On that day we will meet from 220-400 pm. The instructor will question the team members during the presentation.

2. Part 1

We return to the nonlinear systems, f(x) = 0, where f is the steep-valley function and tridiagonal steep-valley function. We will solve for f(x) = 0 (or find a local min) by using the Newton differential equation,

\[(ND) \quad x(t) + [f'(x(t))]^{-1} f(x(t)) = 0, \quad x(0) = x_0,\]

for various initial conditions, including the singularity curves. When \(f'\) is singular, we will need to interpret this equation as well as along the line \(x_1 + x_2 = 0\) (in the single block case). Usually the interpretation will be the generalized inverse. For the single block case, we can work out many of the details analytically. We want to investigate where the integrated curve “goes” for various regions of \(\mathbb{R}^2\), especially for \(k = 1\).

Recall that for \(k = 1\),

\[(F) \quad f = \begin{bmatrix} (x_1 - x_2^7)^3 + x_2^3 \\ (x_1 + x_2)^{1/5} + 1 \end{bmatrix},\]

and

\[(J) \quad f' = \begin{bmatrix} 3(x_1 - x_2^7)^2 & -21x_2^6(x_1 - x_2^7)^2 + 3x_2^{-2} \\ \frac{2}{|x_1 + x_2|^8} & \frac{2}{|x_1 + x_2|^8} \end{bmatrix} = \begin{bmatrix} b & c \\ a & a \end{bmatrix}.\]

Verify that
\[(\text{INV}) \quad f'^{-1} = \frac{1}{a(b-c)}\begin{bmatrix} a & -c \\ -a & b \end{bmatrix}\]

and that \(f' y + f = 0\) for

\[(\text{ND}) \quad y = \frac{1}{a(b-c)}\begin{bmatrix} -af + cf_2 \\ af + bf_2 \end{bmatrix}.\]

This allows us to compute \(g = y = -f'^{-1}f\) for solving \(\dot{x} = g(x(t))\). Along the line \(x_1 + x_2 = 0, a = +\infty\), and the \(y\) can be interpreted as

\[(\text{LN}) \quad y = \frac{1}{(b-c)}\begin{bmatrix} -f_1 \\ f_1 \end{bmatrix},\]

except for the point \(x = [0,0]^T\). On the singularity curves, we have \(b = c\); verify that the normal equations (of least squares) are:

\[(\text{NE}) \quad (a^2 + b^2)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} y = -\begin{bmatrix} bf_1 + af_2 \\ bf_1 + af_2 \end{bmatrix}.\]

We see that \(y_1 + y_2 = \frac{-bf + af_1}{a^2 + b^2}\) so that, if \(r = \frac{-bf_1 + af_1}{a^2 + b^2}\), then \(z = \begin{bmatrix} pr \\ (1-p)r \end{bmatrix}\) is a solution for any real \(p\). Verify that the \(p\) that minimizes the 2-norm of \(z\) is \(p = 1/2\) and that this \(y = z(1/2)\) is the least squares solution of Thm 4, section 2.8.4.

Let \(\dot{h}^2(t) = f^T f\). Then using (ND) we see that \(2\dot{h}h = 2f^T f' \dot{x} = 2f^T f' y\) and

\[(\text{NRM1}) \quad \dot{h} = -h;\]

thus,

\[(\text{NRM2}) \quad h(x(t)) = e^{t} h_0,\]

away from singular \(f'\). When \(f'\) is singular, verify that using the \(y\) associated with (NE) gives

\[(\text{NRM3}) \quad \dot{h} h = \frac{(af_2 + bf_1)^2}{a^2 + b^2} < 0\]

unless \(f'^T f = 0\) (local min or \(f = 0\) condition). This implies that \(-h \leq \dot{h} < 0\) and that \(h\) decays more slowly than indicated by (NRM2). However, unless we stay on a singularity
curve for an interval of $t$, we see that (NRM2) should hold. Verify also that, along the line $x_1 + x_2 = 0$ (except for $x = [0,0]^T$), $\dot{h} = -f_i^2 < 0$ using (LN) (why? note eqn 3.143). Note also that, while it does not seem possible to interpret $r$ associated with (NRM2) at $x = [0,0]^T$, $\dot{h}$ can be given the interpretation, $\dot{h} = -1$, there which seems peculiar. However, along the singularity curves, we have $y_1 = y_2$ which means that $\frac{dx_1}{dx_2} = 1$. If we use this interpretation for the point $x = [0,0]^T$, we could just move incrementally along the line $x_1 = x_2$ to get “started” and maybe this explains the $\dot{h} = -1$ limiting behavior. Please examine this.

3. Part 2

We will solve the 2-point boundary value problem:

\[(BVP) \quad -j(x,u,u') + u = 100e^{-64(x-3/4)^2}, \quad x \in [0,1], \quad (\quad = d/dx ) \]
\[j = u' + \beta u, \]

with boundary conditions,

\[(BC) \quad u(0) = u(1) = 0.\]

We will compare the “finite difference” discretization of section 4.5.2 with the “constant $j$” discretization of section 4.6.4. They will be the same when $\beta = 0$. Both will lead to solving tridiagonal systems of linear equation, $Tu = b$. DO NOT use full matrix equations! The vector $u$ will have $n$ components that will approximate the solution to the BVP; that is, $u_i$ will approximate $u(x_i)$ for $x_i \in (0,1)$.

**Specification**

[A] First take $\beta = 0$, and use equally spaced gridpoints to get a feel for the problem and solution. Both discretizations should be the same.

[B] Now try $\beta = \pm 10, \pm 100, \pm 500$. Start with equally spaced points. Notice that the finite difference discretization requires smaller grid spacing. Look up Richardson extrapolation for some ideas on how to get a truncation error estimate for constant-$j$ (since $j$ is constant the error term should be $O(h)$). We will also discuss this during the hw4 special classes.

[C] Finally make everything efficient in [B] by using nonuniformly spaced points (for constant-$j$). How few flops give a good answer?
Betsy working on HW 4