Which of the following languages are CFL?

- L=\{a^n b^n c^j \mid 0 < n \leq j\}
- L=\{a^n b^i a^n b^j \mid n > 0, j > 0\}
- L=\{a^n b^i a^k b^p \mid n + j \leq k + p, n > 0, j > 0, k > 0, p > 0\}

**Pumping Lemma for Regular Language’s:** Let L be a regular language. Then there is a constant m such that w \in L, |w| \geq m, w = xyz such that

- |xy| \leq m
- |y| \geq 1
- for all i \geq 0, x y^i z \in L

**Pumping Lemma for CFL’s** Let L be any infinite CFL. Then there is a constant m depending only on L, such that for every string w in L, with |w| \geq m, we may partition w = uvxyz such that:

|vxy| \leq m, (limit on size of substring)
|vy| \geq 1, (v and y not both empty)
For all i \geq 0, u v^i x y^i z \in L

**Proof:** (sketch) There is a CFG G s.t. L=L(G).

Consider the parse tree of a long string in L.

For any long string, some nonterminal N must appear twice in the path.
Example: Consider \( L = \{ a^n b^n c^n : n \geq 1 \} \). Show \( L \) is not a CFL.

- **Proof:** (by contradiction)

  Assume \( L \) is a CFL and apply the pumping lemma.

  Let \( m \) be the constant in the pumping lemma and consider \( w = a^m b^m c^m \). Note \(|w| \geq m\).

  Show there is no division of \( w \) into \( uvxyz \) such that 
  \[ |vy| \geq 1, |vxy| \leq m, \text{ and } uv^i xy^i z \in L \text{ for } i = 0, 1, 2, \ldots \]

  Case 1: Neither \( v \) nor \( y \) can contain 2 or more distinct symbols. If \( v \) contains \( a \)'s and \( b \)'s, then 
  \( uv^2 x y^2 z \notin L \) since \( t_1 + t_2 > 0, n(b) > n(a) \)'s.

  Thus, \( v \) and \( y \) can be only \( a \)'s, \( b \)'s, or \( c \)'s (not mixed).

  Case 2: \( v = a^{t_1} \), then \( y = a^{t_2} \) or \( b^{t_3} \) (|\( vxy \)| \leq \( m \))

  If \( y = a^{t_2} \), then \( uv^2 x y^2 z = a^{m+t_1+t_2} b^m c^m \notin L \) since \( t_1 + t_2 > 0, n(a) > n(b) \)’s (number of \( a \)'s is greater than number of \( b \)'s)

  If \( y = b^{t_3} \), then \( uv^2 x y^2 z = a^{m+t_1} b^{m+t_3} c^m \notin L \) since \( t_1 + t_3 > 0, n(a) > n(c) \)’s or \( n(b) > n(c) \)’s.

  Case 3: \( v = b^{t_1} \), then \( y = b^{t_2} \) or \( c^{t_3} \)

  If \( y = b^{t_2} \), then \( uv^2 x y^2 z = a^m b^{m+t_1+t_2} c^m \notin L \) since \( t_1 + t_2 > 0, n(b) > n(a) \)’s.

  If \( y = c^{t_3} \), then \( uv^2 x y^2 z = a^m b^m c^{m+t_3} \notin L \) since \( t_1 + t_3 > 0, n(b) > n(c) \)’s or \( n(c) > n(a) \)’s.

  Case 4: \( v = c^{t_1} \), then \( y = c^{t_2} \)

  then, \( uv^2 x y^2 z = a^m b^m c^{m+t_1+t_2} \notin L \) since \( t_1 + t_2 > 0, n(c) > n(a) \)’s.

  Thus, there is no breakdown of \( w \) into \( uvxyz \) such that \(|vy| \geq 1, |vxy| \leq m \) and for all \( i \geq 0, uv^i xy^i z \) is in \( L \). Contradiction, thus, \( L \) is not a CFL. Q.E.D.
Example Why would we want to recognize a language of the type \( \{a^n b^n c^n : n \geq 1 \} \)?

Example: Consider \( L = \{a^n b^n c^p : p > n > 0 \} \). Show \( L \) is not a CFL.

Proof: Assume \( L \) is a CFL and apply the pumping lemma. Let \( m \) be the constant in the pumping lemma and consider \( w = \). Note \(|w| \geq m\).

Show there is no division of \( w \) into \( uvxyz \) such that \(|vy| \geq 1\), \(|vxy| \leq m\), and \( uv^i xy^i z \in L \) for \( i = 0, 1, 2, \ldots \).

Thus, there is no breakdown of \( w \) into \( uvxyz \) such that \(|vy| \geq 1\), \(|vxy| \leq m\) and for all \( i \geq 0\), \( uv^i xy^i z \) is in \( L \). Contradiction, thus, \( L \) is not a CFL. Q.E.D.
Example: Consider $L = \{a^j b^k : k = j^2\}$. Show $L$ is not a CFL.

- **Proof:** Assume $L$ is a CFL and apply the pumping lemma. Let $m$ be the constant in the pumping lemma and consider $w = \text{_________}$

Show there is no division of $w$ into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$, and $uv^i xy^i z \in L$ for $i = 0, 1, 2, \ldots$.

Case 1: Neither $v$ nor $y$ can contain 2 or more distinct symbols. If $v$ contains $a$’s and $b$’s, then $uv^2 xy^2 z \notin L$ since there will be $b$’s before $a$’s.

Thus, $v$ and $y$ can be only $a$’s, and $b$’s (not mixed).

Thus, there is no breakdown of $w$ into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$ and for all $i \geq 0$, $uv^i xy^i z$ is in $L$. Contradiction, thus, $L$ is not a CFL. Q.E.D.

**Exercise:** Prove the following is not a CFL by applying the pumping lemma. (answer is at the end of this handout).

Consider $L = \{a^{2n} b^{3p} c^n d^p : n, p \geq 0\}$. Show $L$ is not a CFL.
**Example:** Consider $L = \{ \overline{ww} : w \in \Sigma^* \}$, $\Sigma = \{a, b\}$, where $\overline{w}$ is the string $w$ with each occurrence of $a$ replaced by $b$ and each occurrence of $b$ replaced by $a$. For example, $w = baaa$, $\overline{w} = abbb$, $\overline{w}w = baaaabbb$. Show $L$ is not a CFL.

- **Proof:** Assume $L$ is a CFL and apply the pumping lemma. Let $m$ be the constant in the pumping lemma and consider $w = \_\_\_\_\_\_\_\_\_$. Show there is no division of $w$ into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$, and $uv^ixy^iz \in L$ for $i = 0, 1, 2, \ldots$

Thus, there is no breakdown of $w$ into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$ and for all $i \geq 0$, $uv^ixy^iz \in L$. Contradiction, thus, $L$ is not a CFL. Q.E.D.
**Example:** Consider $L = \{a^n b^n b^m a^n\}$. $L$ is a CFL. The pumping lemma should apply!

Let $m \geq 4$ be the constant in the pumping lemma. Consider $w = a^m b^m b^m a^m$.

We can break $w$ into $uvxyz$, with:

If you apply the pumping lemma to a CFL, then you should find a partition of $w$ that works!

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** Chap 8.2 Closure Properties of CFL’s **

**Theorem** CFL’s are closed under union, concatenation, and star-closure.

- **Proof:**

  Given 2 CFG $G_1 = (V_1, T_1, S_1, P_1)$ and $G_2 = (V_2, T_2, S_2, P_2)$

  - **Union:**
    Construct $G_3$ s.t. $L(G_3) = L(G_1) \cup L(G_2)$.
    $G_3 = (V_3, T_3, S_3, P_3)$

  - **Concatenation:**
    Construct $G_3$ s.t. $L(G_3) = L(G_1) \circ L(G_2)$.
    $G_3 = (V_3, T_3, S_3, P_3)$
– Star-Closure
  Construct $G_3$ s.t. $L(G_3) = L(G_1)^*$
  $G_3 = (V_3, T_3, S_3, P_3)$

QED.

**Theorem** CFL’s are NOT closed under intersection and complementation.

• **Proof:**
  – Intersection:

  – Complementation:
Theorem: CFL's are closed under regular intersection. If $L_1$ is CFL and $L_2$ is regular, then $L_1 \cap L_2$ is CFL.

• Proof: (sketch) This proof is similar to the construction proof in which we showed regular languages are closed under intersection. We take a NPDA for $L_1$ and a DFA for $L_2$ and construct a NPDA for $L_1 \cap L_2$.

$M_1 = (Q_1, \Sigma, \Gamma, \delta_1, q_0, z, F_1)$ is an NPDA such that $L(M_1) = L_1$.

$M_2 = (Q_2, \Sigma, \delta_2, q'_0, F_2)$ is a DFA such that $L(M_2) = L_2$.

Example of replacing arcs (NOT a Proof!):
Note this is not a proof, but sketches how we will combine the DFA and NPDA. We must formally define $\delta_3$. If

then

Must show

if and only if

Must show:

$w \in L(M_3)$ iff $w \in L(M_1)$ and $w \in L(M_2)$.

QED.
Questions about CFL:

1. Decide if CFL is empty?
2. Decide if CFL is infinite?

Example: Consider $L = \{a^{2n}b^{2m}c^nd^m : n, m \geq 0\}$. Show $L$ is not a CFL.

- **Proof:** Assume $L$ is a CFL and apply the pumping lemma. Let $m$ be the constant in the pumping lemma and consider $w = a^{2m}b^{2m}c^md^m$.

Show there is no division of $w$ into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$, and $uv^ixy^iz \in L$ for $i = 0, 1, 2, \ldots$.

**Case 1:** Neither $v$ nor $y$ can contain 2 or more distinct symbols. If $v$ contains $a$'s and $b$'s, then $uv^2xy^2z \notin L$ since there will be $b$'s before $a$'s.

Thus, $v$ and $y$ can be only $a$'s, $b$'s, $c$'s, or $d$'s (not mixed).

**Case 2:** $v = a^{t_1}$, then $y = a^{t_2}$ or $b^{t_3}$ ($|vxy| \leq m$)

If $y = a^{t_2}$, then $uv^2xy^2z = a^{2m+t_1+t_2}b^{2m}c^{m}d^m \notin L$ since $t_1 + t_2 > 0$, the number of $a$'s is not twice the number of $c$'s.

If $y = b^{t_3}$, then $uv^2xy^2z = a^{2m+t_1+t_2}b^{2m}c^{m}d^m \notin L$ since $t_1 + t_3 > 0$, either the number of $a$'s

(denoted $n(a)$) is not twice $n(c)$ or $n(b)$ is not twice $n(d)$.

**Case 3:** $v = b^{t_1}$, then $y = b^{t_2}$ or $c^{t_3}$

If $y = b^{t_2}$, then $uv^2xy^2z = a^{2m}b^{2m+t_1+t_2}c^md^m \notin L$ since $t_1 + t_2 > 0$, $n(b) > 2*n(d)$.

If $y = c^{t_3}$, then $uv^2xy^2z = a^{2m}b^{2m+t_1}c^{m+t_3}d^m \notin L$ since $t_1 + t_3 > 0$, either $n(b) > 2*n(d)$ or

$2*n(c) > n(a)$.

**Case 4:** $v = c^{t_1}$, then $y = c^{t_2}$ or $d^{t_3}$

If $y = c^{t_2}$, then $uv^2xy^2z = a^{2m}b^{2m}c^{m+t_1+t_2}d^m \notin L$ since $t_1 + t_2 > 0$, $2*n(c) > n(a)$.

If $y = d^{t_3}$, then $uv^2xy^2z = a^{2m}b^{2m}c^{m+t_1}d^{m+t_3} \notin L$ since $t_1 + t_3 > 0$, either $2*n(c) > n(a)$ or

$2*n(d) > n(b)$.

**Case 5:** $v = d^{t_1}$, then $y = d^{t_2}$

then $uv^2xy^2z = a^{2m}b^{2m}c^{m}d^{m+t_1+t_2} \notin L$ since $t_1 + t_2 > 0$, $2*n(d) > n(c)$.

Thus, there is no breakdown of $w$ into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$ and for all $i \geq 0$, $uv^i xy^i z$ is in $L$. Contradiction, thus, $L$ is not a CFL. Q.E.D.