Section: Decidability

Computability A function $f$ with domain $D$ is *computable* if there exists some TM $M$ such that $M$ computes $f$ for all values in its domain.

Decidability A problem is *decidable* if there exists a TM that can answer yes or no to every statement in the domain of the problem.
The Halting Problem

Domain: set of all TMs and all strings \( w \).

Question: Given coding of \( M \) and \( w \), does \( M \) halt on \( w \)?
Theorem The halting problem is undecidable.

Proof: (by contradiction)

- Assume there is a TM $H$ (or algorithm) that solves this problem. TM $H$ has 2 final states, $q_y$ represents yes and $q_n$ represents no.

$$ H(w_M, w) = \begin{cases} 
\text{halts } q_y \text{ if } M \text{ halts on } w \\
\text{halts } q_n \text{ if } M \text{ doesn’t halt on } w
\end{cases} $$

TM $H$ always halts in a final state.
Construct TM $H'$ from $H$

$$H'(w_M, w) = \begin{cases} \text{halts} & \text{if } M \text{ not halt on } w \\ \text{not halt} & \text{if } M \text{ halts on } w \end{cases}$$

Construct TM $\hat{H}$ from $H'$

$$\hat{H}(w_M) = \begin{cases} \text{halts} & \text{if } M \text{ not halt on } w_M \\ \text{not halt} & \text{if } M \text{ halts on } w_M \end{cases}$$

Note that $\hat{H}$ is a TM.

There is some encoding of it, say $\hat{w}_{\hat{H}}$.

What happens if we run $\hat{H}$ with input $\hat{w}_{\hat{H}}$?
Theorem If the halting problem were decidable, then every recursively enumerable language would be recursive. Thus, the halting problem is undecidable.

- **Proof:** Let \( L \) be an RE language over \( \Sigma \).
  Let \( M \) be the TM such that \( L = L(M) \).
  Let \( H \) be the TM that solves the halting problem.
A problem A is *reduced* to problem B if the decidability of B follows from the decidability of A. Then if we know B is undecidable, then A must be undecidable.
State-entry problem Given TM
\( M = (Q, \Sigma, \Gamma, \delta, q_0, B, F) \), state \( q \in Q \), and
string \( w \in \Sigma^* \), is state \( q \) ever entered
when \( M \) is applied to \( w \)?

This is an undecidable problem!

- Proof:
  
  TM \( E \) solves state-entry problem
  
  \[
  E'(w_M, w) = \begin{cases} 
  M \text{ halts on } w & \text{if } ? \\
  M \text{ doesn’t halt on } w & \text{if } ? 
  \end{cases}
  \]