Section: Pushdown Automata

Ch. 7 - Pushdown Automata

A DFA = (Q, Σ, δ, q₀, F)

input tape

|   | a | a | b | b | a | b |   |

tape head  head moves

current state

0 1
Modify DFA by adding a stack. New machine is called Pushdown Automata (PDA).
Definition: Nondeterministic PDA (NPDA) is defined by

\[ M = (Q, \Sigma, \Gamma, \delta, q_0, z, F) \]

where

- \( Q \) is finite set of states
- \( \Sigma \) is tape (input) alphabet
- \( \Gamma \) is stack alphabet
- \( q_0 \) is initial state
- \( z \) is start stack symbol, \( z \in \Gamma \)
- \( F \subseteq Q \) is set of final states.

\( \delta : Q \times (\Sigma \cup \{\lambda\}) \times \Gamma \rightarrow \) finite subsets of \( Q \times \Gamma^* \)
Example of transitions

\[ \delta(q_1,a,b) = \{(q_3,b),(q_4,ab),(q_6,\lambda)\} \]

The diagram for the above transitions is:
Instantaneous Description:

\[(q,w,u)\]

Description of a Move:

\[(q_1,aw,bx) \vdash (q_2,w,yx)\]

iff

Definition Let \( M=(Q,\Sigma,\Gamma,\delta,q_0,z,F) \) be a NPDA. \( L(M)=\{w \in \Sigma^* \mid (q_0,w,z) \vdash^* (p,\lambda,u), \ p \in F, \ u \in \Gamma^*\} \). The NPDA accepts all strings that start in \( q_0 \) and end in a final state.
Example: \( L = \{ a^n b^n | n \geq 0 \} \), \( \Sigma = \{ a, b \} \),
\( \Gamma = \{ z, a \} \)
Another Definition for Language Acceptance

NPDA $M$ accepts $L(M)$ by empty stack:

$$L(M) = \{ w \in \Sigma^* \mid (q_0, w, z)^* \vdash (p, \lambda, \lambda) \}$$
Example: \( L = \{ a^n b^m c^{n+m} \mid n, m > 0 \} \),
\( \Sigma = \{ a, b, c \} \), \( \Gamma = \{ 0, z \} \)
Example: \( L = \{ ww^R | w \in \Sigma^+ \} \), \( \Sigma = \{ a, b \} \),
\( \Gamma = \{ z, a, b \} \)
Example: \( L=\{ww \mid w \in \Sigma^*\}, \Sigma = \{a, b\} \)

Examples for you to try on your own: (solutions are at the end of the handout).

- \( L=\{a^n b^m \mid m > n, m, n > 0\}, \Sigma = \{a, b\}, \Gamma = \{z, a\} \)
- \( L=\{a^n b^{n+m} c^m \mid n, m > 0\}, \Sigma = \{a, b, c\} \)
- \( L=\{a^n b^{2n} \mid n > 0\}, \Sigma = \{a, b\} \)
Definition: A PDA
$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$ is deterministic if
for every $q \in Q$, $a \in \Sigma \cup \{\lambda\}$, $b \in \Gamma$

1. $\delta(q, a, b)$ contains at most 1 element
2. if $\delta(q, \lambda, b) \neq \emptyset$ then $\delta(q, c, b) = \emptyset$ for all $c \in \Sigma$

Definition: $L$ is DCFL iff $\exists$ DPDA $M$
s.t. $L = L(M)$. 
Examples:

1. Previous pda for \( \{a^n b^n | n \geq 0\} \) is deterministic?

2. Previous pda for 
\( \{a^n b^m c^{n+m} | n, m > 0\} \) is deterministic?

3. Previous pda for 
\( \{ww^R | w \in \Sigma^+\}, \Sigma = \{a, b\} \) is deterministic?
Example: \( L = \{ a^n b^m | m > n, m, n > 0 \} \), \( \Sigma = \{ a, b \} \), \( \Gamma = \{ z, a \} \)

\[
\begin{array}{c}
q_0 \xrightarrow{a,z;az} q_1 \xrightarrow{a,a;aa} b,a;\lambda \xrightarrow{b,z;z} q_3
\end{array}
\]

Example: \( L = \{ a^n b^{n+m} c^m | n, m > 0 \} \), \( \Sigma = \{ a, b, c \} \),

\[
\begin{array}{c}
q_0 \xrightarrow{a,z;az} q_1 \xrightarrow{a,a;aa} b,a; \xrightarrow{b,z;bz} q_3
\end{array}
\]

Example: \( L = \{ a^n b^{2n} | n > 0 \} \), \( \Sigma = \{ a, b \} \)

\[
\begin{array}{c}
q_0 \xrightarrow{a,z;aa} q_1 \xrightarrow{b,a;\lambda} q_2 \xrightarrow{\lambda,z;\lambda} q_3
\end{array}
\]
Chapter 7.2

Theorem Given NPDA M that accepts by final state, \( \exists \) NPDA M\(^\prime\) that accepts by empty stack s.t. \( \mathcal{L}(M) = \mathcal{L}(M') \).

- Proof (sketch)
  
  \[ M = (Q, \Sigma, \Gamma, \delta, q_0, z, F) \]
  
  Construct \( M' = (Q', \Sigma, \Gamma', \delta', q_s, z', F') \)
Theorem: Given NPDA $M$ that accepts by empty stack, $\exists$ NPDA $M'$ that accepts by final state.

Proof: (sketch)

$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$

Construct $M' = (Q', \Sigma, \Gamma', \delta', q_s, z', F')$
Theorem For any CFL $L$ not containing $\lambda$, $\exists$ an NPDA $M$ s.t. $L=L(M)$.

- Proof (sketch)
  
  Given ($\lambda$-free) CFL $L$.
  
  $\Rightarrow \exists$ CFG $G$ such that $L=L(G)$.
  
  $\Rightarrow \exists G'$ in GNF, s.t. $L(G)=L(G')$.
  
  $G'=(V,T,S,P)$. All productions in $P$ are of the form:
Example: Let $G' = (V, T, S, P)$, $P =$

\[
\begin{align*}
S & \rightarrow aSA \mid aAA \mid b \\
A & \rightarrow bBBB \\
B & \rightarrow b
\end{align*}
\]
Theorem Given a NPDA $M$, $\exists$ a NPDA $M'$ s.t. all transitions have the form $\delta(q_i,a,A) = \{c_1, c_2, \ldots c_n\}$ where

$$c_i = (q_j, \lambda)$$

or

$$c_i = (q_j, BC)$$

Each move either increases or decreases stack contents by a single symbol.

- Proof (sketch)
Theorem If $L=L(M)$ for some NPDA $M$, then $L$ is a CFL.

• Proof: Given NPDA $M$.
First, construct an equivalent NPDA $M'$ that will be easier to work with. Construct $M'$ such that

1. accepts if stack is empty
2. each move increases or decreases stack content by a single symbol. (can only push 2 variables or no variables with each transition)

$M' = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$

Construct $G = (V, \Sigma, S, P)$ where

$V = \{(q_i c q_j) | q_i, q_j \in Q, c \in \Gamma\}$

Goal: $(q_0 z q_f)$ which will be the start symbol in the grammar.
Example:

$L(M) = \{aa^*b\}$, $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F),$

$Q = \{q_0, q_1, q_2, q_3\},$

$\Sigma = \{a, b\}, \Gamma = \{A, z\}, F = \{\}.}$
Construct the grammar $G=(V,T,S,P)$,

$V=\{(q_0Aq_0), (q_0zq_0), (q_0Aq_1), (q_0zq_1), \ldots \}$

$T=\Sigma$

$S=(q_0zq_2)$

$P=$
From transition 1 \((q_0 A q_1) \rightarrow b\)

From transition 2 \((q_1 z q_2) \rightarrow \lambda\)

From transition 3 \((q_0 A q_3) \rightarrow a\)

From transition 4 \((q_0 z q_0) \rightarrow a(q_0 A q_0)(q_0 z q_0)\) | 
\(a(q_0 A q_1)(q_1 z q_0)\) | 
\(a(q_0 A q_2)(q_2 z q_0)\) | 
\(a(q_0 A q_3)(q_3 z q_0)\) 

\((q_0 z q_1) \rightarrow a(q_0 A q_0)(q_0 z q_1)\) | 
\(a(q_0 A q_1)(q_1 z q_1)\) | 
\(a(q_0 A q_2)(q_2 z q_1)\) | 
\(a(q_0 A q_3)(q_3 z q_1)\) 

\((q_0 z q_2) \rightarrow a(q_0 A q_0)(q_0 z q_2)\) | 
\(a(q_0 A q_1)(q_1 z q_2)\) | 
\(a(q_0 A q_2)(q_2 z q_2)\) | 
\(a(q_0 A q_3)(q_3 z q_2)\) 

\((q_0 z q_3) \rightarrow a(q_0 A q_0)(q_0 z q_3)\) | 
\(a(q_0 A q_1)(q_1 z q_3)\) | 
\(a(q_0 A q_2)(q_2 z q_3)\) | 
\(a(q_0 A q_3)(q_3 z q_3)\)
From transition 5 \((q_3 z q_0) \rightarrow (q_0 A q_0)(q_0 z q_0)\) |
\((q_0 A q_1)(q_1 z q_0)\) |
\((q_0 A q_2)(q_2 z q_0)\) |
\((q_0 A q_3)(q_3 z q_0)\)

\((q_3 z q_1) \rightarrow (q_0 A q_0)(q_0 z q_1)\) |
\((q_0 A q_1)(q_1 z q_1)\) |
\((q_0 A q_2)(q_2 z q_1)\) |
\((q_0 A q_3)(q_3 z q_1)\)

\((q_3 z q_2) \rightarrow (q_0 A q_0)(q_0 z q_2)\) |
\((q_0 A q_1)(q_1 z q_2)\) |
\((q_0 A q_2)(q_2 z q_2)\) |
\((q_0 A q_3)(q_3 z q_2)\)

\((q_3 z q_3) \rightarrow (q_0 A q_0)(q_0 z q_3)\) |
\((q_0 A q_1)(q_1 z q_3)\) |
\((q_0 A q_2)(q_2 z q_3)\) |
\((q_0 A q_3)(q_3 z q_3)\)
Recognizing aaab in M:

\[(q_0, aaab, z) \vdash (q_0, aab, Az)\]
\[\vdash (q_3, ab, z)\]
\[\vdash (q_0, ab, Az)\]
\[\vdash (q_3, b, z)\]
\[\vdash (q_0, b, Az)\]
\[\vdash (q_1, \lambda, z)\]
\[\vdash (q_2, \lambda, \lambda)\]

Derivation of string aaab in G:

\[(q_0zq_2) \Rightarrow a(q_0Aq_3)(q_3zq_2)\]
\[\Rightarrow aa(q_3zq_2)\]
\[\Rightarrow aa(a(q_0Aq_3)(q_3zq_2))\]
\[\Rightarrow aaa(q_3zq_2)\]
\[\Rightarrow aaa(a(q_0Aq_1)(q_1zq_2))\]
\[\Rightarrow aaab(q_1zq_2)\]
\[\Rightarrow aaab\]
Chapter 7.3

Definition: A PDA

\( M = (Q, \Sigma, \Gamma, \delta, q_0, z, F) \) is deterministic if for every \( q \in Q, \ a \in \Sigma \cup \{\lambda\}, \ b \in \Gamma \)

1. \( \delta(q, a, b) \) contains at most 1 element
2. if \( \delta(q, \lambda, b) \neq \emptyset \) then \( \delta(q, c, b) = \emptyset \) for all \( c \in \Sigma \)

Definition: \( L \) is DCFL iff \( \exists \) DPDA \( M \) s.t. \( L = L(M) \).
Examples:

1. Previous pda for \( \{a^nb^n \mid n \geq 0\} \) is deterministic.

2. Previous pda for \( \{a^nb^mc^{n+m} \mid n, m > 0\} \) is deterministic.

3. Previous pda for \( \{ww^R \mid w \in \Sigma^+\}, \Sigma = \{a, b\} \) is nondeterministic.

Note: There are CFL’s that are not deterministic.

\[ L = \{a^nb^n \mid n \geq 1\} \cup \{a^nb^{2n} \mid n \geq 1\} \text{ is a CFL and not a DCFL.} \]

• Proof:

\[ L = \{a^nb^n : n \geq 1\} \cup \{a^nb^{2n} : n \geq 1\} \]

It is easy to construct a NPDA for \( \{a^nb^n : n \geq 1\} \) and a NPDA for \( \{a^nb^{2n} : n \geq 1\} \). These two can be joined together by a new start state
and $\lambda$-transitions to create a NPDA for $L$. Thus, $L$ is CFL.

Now show $L$ is not a DCFL. Assume that there is a deterministic PDA $M$ such that $L = L(M)$. We will construct a PDA that recognizes a language that is not a CFL and derive a contradiction.

Construct a PDA $M'$ as follows:

1. Create two copies of $M$: $M_1$ and $M_2$. The same state in $M_1$ and $M_2$ are called cousins.

2. Remove accept status from accept states in $M_1$, remove initial status from initial state in $M_2$. In our new PDA, we will start in $M_1$ and accept in $M_2$.

3. Outgoing arcs from old accept states in $M_1$, change to end up in the cousin of its destination in
$M_2$. This joins $M_1$ and $M_2$ into one PDA. There must be an outgoing arc since you must recognize both $a^nb^n$ and $a^nb^{2n}$. After reading $n$ $b$’s, must accept if no more $b$’s and continue if there are more $b$’s.

4. Modify all transitions that read a $b$ and have their destinations in $M_2$ to read a $c$.

This is the construction of our new PDA.

When we read $a^nb^n$ and end up in an old accept state in $M_1$, then we will transfer to $M_2$ and read the rest of $a^nb^{2n}$. Only the $b$’s in $M_2$ have been replaced by $c$’s, so the new machine accepts $a^nb^nc^n$.

The language accepted by our new PDA is $a^nb^nc^n$. But this is not a CFL. Contradiction! Thus there is
no deterministic PDA $M$ such that $L(M) = L$. Q.E.D.