Regular Expressions

Method to represent strings in a language

+ union (or)
◦ concatenation (AND) (can omit)
∗ star-closure (repeat 0 or more times)

Example:

\[(a + b)^* \circ a \circ (a + b)^*\]

Example:

\[(aa)^*\]

Definition Given \(\Sigma\),

1. \(\emptyset, \lambda, a \in \Sigma\) are R.E.
2. If \(r\) and \(s\) are R.E. then
   - \(r+s\) is R.E.
   - \(rs\) is R.E.
   - \((r)\) is a R.E.
   - \(r^*\) is R.E.
3. \(r\) is a R.E. iff it can be derived from (1) with a finite number of applications of (2).

Definition: \(L(r)\) = language denoted by R.E. \(r\).

1. \(\emptyset, \{\lambda\}, \{a\}\) are L denoted by a R.E.
2. if \(r\) and \(s\) are R.E. then
   - \(L(r+s) = L(r) \cup L(s)\)
   - \(L(rs) = L(r) \circ L(s)\)
   - \(L((r)) = L(r)\)
   - \(L((r)^*) = (L(r)^*)\)

Precedence Rules

\(\ast\) highest
\(\circ\)
\(+\)

Example:

\(ab^* + c =\)
Examples:

1. \( \Sigma = \{a, b\}, \{w \in \Sigma^* \mid w \text{ has an odd number of } a\text{'s followed by an even number of } b\text{'s}\} \).

2. \( \Sigma = \{a, b\}, \{w \in \Sigma^* \mid w \text{ has no more than 3 } a\text{'s and must end in } ab\} \).

3. Regular expression for positive and negative integers

Section 3.2 Equivalence of DFA and R.E.

**Theorem** Let \( r \) be a R.E. Then \( \exists \) NFA \( M \) s.t. \( L(M) = L(r) \).

- **Proof:**
  - \( \emptyset \)
  - \( \{\lambda\} \)
  - \( \{a\} \)
  
  Suppose \( r \) and \( s \) are R.E.

  1. \( r + s \)
  2. \( r \circ s \)
  3. \( r^* \)

**Example**

\( ab^* + c \)

**Theorem** Let \( L \) be regular. Then \( \exists \) R.E. \( r \) s.t. \( L=L(r) \).

Proof Idea: remove states successively, generating equivalent generalized transition graphs (GTG) until only two states are left (one initial state and one final state).

- **Proof:**
  - \( L \) is regular
    - \( \Rightarrow \exists \)
    1. Assume \( M \) has one final state and \( q_0 \not\in F \)
    2. Convert to a generalized transition graph (GTG), all possible edges are present.
      
      If no edge, label with
      
      Let \( r_{ij} \) stand for label of the edge from \( q_i \) to \( q_j \)

      3. If the GTG has only two states, then it has the following form:
      
      In this case the regular expression is:
      
      \( r = (r_{ii}^* r_{ij} r_{jj}^* r_{ji})^* r_{ii}^* r_{ij} r_{jj}^* \)

      4. If the GTG has three states then it must have the following form:
In this case, make the following replacements:

<table>
<thead>
<tr>
<th>REPLACE</th>
<th>WITH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_{ii}$</td>
<td>$r_{ii} + r_{ik}r_{kk}^*r_{ki}$</td>
</tr>
<tr>
<td>$r_{jj}$</td>
<td>$r_{jj} + r_{jk}r_{kk}^*r_{kj}$</td>
</tr>
<tr>
<td>$r_{ij}$</td>
<td>$r_{ij} + r_{ik}r_{kk}^*r_{kj}$</td>
</tr>
<tr>
<td>$r_{ji}$</td>
<td>$r_{ji} + r_{jk}r_{kk}^*r_{ki}$</td>
</tr>
</tbody>
</table>

After these replacements, remove state $q_k$ and its edges.

5. If the GTG has four or more states, pick a state $q_k$ to be removed (not initial or final state).

For all $o \neq k, p \neq k$ use the rule

$r_{op}$ replaced with $r_{op} + r_{ok}r_{kk}^*r_{kp}$

with different values of $o$ and $p$.

When done, remove $q_k$ and all its edges. Continue eliminating states until only two states are left.

Finish with step 3.

6. In each step, simplify the regular expressions $r$ and $s$ with:
\[
\begin{align*}
    r + r &= r \\
    s + r^* s &= \\
    r + \emptyset &= \\
    r \emptyset &= \\
    \emptyset^* &= \\
    r \lambda &= \\
    (\lambda + r)^* &= \\
    (\lambda + r)^* &= \\
\end{align*}
\]
and similar rules.

Example:

Section 3.3

Grammar \( G = (V, T, S, P) \)

- **V** variables (nonterminals)
- **T** terminals
- **S** start symbol
- **P** productions

**Right-linear grammar:**

all productions of form

\[
A \rightarrow xB \\
A \rightarrow x
\]

where \( A, B \in V, \ x \in T^* \)

**Left-linear grammar:**

all productions of form

\[
A \rightarrow Bx \\
A \rightarrow x
\]

where \( A, B \in V, \ x \in T^* \)

**Definition:**

A regular grammar is a right-linear or left-linear grammar.
Example 1:

\[ G = (\{S\}, \{a,b\}, S, P), \quad P = \]
\[ S \rightarrow abS \]
\[ S \rightarrow \lambda \]
\[ S \rightarrow Sab \]

Example 2:

\[ G = (\{S,B\}, \{a,b\}, S, P), \quad P = \]
\[ S \rightarrow aB \mid bS \mid \lambda \]
\[ B \rightarrow aS \mid bB \]

**Theorem:** \( L \) is a regular language iff \( \exists \) regular grammar \( G \) s.t. \( L = L(G) \).

**Outline of proof:**

\( \iff \) Given a regular grammar \( G \)
- Construct NFA \( M \)
- Show \( L(G) = L(M) \)

\( \implies \) Given a regular language
- \( \exists \) DFA \( M \) s.t. \( L = L(M) \)
- Construct reg. grammar \( G \)
- Show \( L(G) = L(M) \)

**Proof of Theorem:**

\( \iff \) Given a regular grammar \( G \)
\[ G = (V, T, S, P) \]
\[ V = \{V_0, V_1, \ldots, V_y\} \]
\[ T = \{v_0, v_1, \ldots, v_z\} \]
\[ S = V_0 \]
- Assume \( G \) is right-linear
  (see book for left-linear case).
- Construct NFA \( M \) s.t. \( L(G) = L(M) \)
- If \( w \in L(G) \), \( w = v_1 v_2 \ldots v_k \)

\[ M = (V \cup \{V_f\}, T, \delta, V_0, \{V_f\}) \]
- \( V_0 \) is the start (initial) state
- For each production, \( V_i \rightarrow aV_j \),
For each production, \( V_i \rightarrow a \),

Show \( L(G) = L(M) \)

Thus, given R.G. G,

\( L(G) \) is regular

(\( \Rightarrow \)) Given a regular language \( L \)

\( \exists \) DFA M s.t. \( L = L(M) \)

\( M = (Q, \Sigma, \delta, q_0, F) \)

\( Q = \{ q_0, q_1, \ldots, q_n \} \)

\( \Sigma = \{ a_1, a_2, \ldots, a_m \} \)

Construct R.G. G s.t. \( L(G) = L(M) \)

\( G = (Q, \Sigma, q_0, P) \)

if \( \delta(q_i, a_j) = q_k \) then

if \( q_k \in F \) then

Show \( w \in L(M) \iff w \in L(G) \)

Thus, \( L(G) = L(M) \).

QED.

Example

\( G = (\{ S, B \}, \{ a, b \}, S, P), P = \)

\( S \rightarrow aB \mid bS \mid \lambda \)

\( B \rightarrow aS \mid bB \)

Example:

\[ q_0 \rightarrow a \rightarrow q_1 \rightarrow b \rightarrow q_0 \]