Section: Properties of Regular Languages

Example

\[ L = \{a^n ba^n \mid n > 0\} \]

Closure Properties

A set is closed over an operation if

\[ L_1, L_2 \in \text{class} \]
\[ L_1 \text{ op } L_2 = L_3 \]
\[ \Rightarrow L_3 \in \text{class} \]
\[ L_1 = \{ x \mid x \text{ is a positive even integer} \} \]

L is closed under

addition?
multiplication?
subtraction?
division?

Closure of Regular Languages

Theorem 4.1 If \( L_1 \) and \( L_2 \) are regular languages, then

\[ L_1 \cup L_2 \]
\[ L_1 \cap L_2 \]
\[ L_1 L_2 \]
\[ \bar{L}_1 \]
\[ L_1^* \]

are regular languages.
Proof(sketch)

$L_1$ and $L_2$ are regular languages
$\Rightarrow \exists$ reg. expr. $r_1$ and $r_2$ s.t.
$L_1 = L(r_1)$ and $L_2 = L(r_2)$
$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$
$\Rightarrow$ closed under union
$r_1 r_2$ is r.e. denoting $L_1 L_2$
$\Rightarrow$ closed under concatenation
$r_1^*$ is r.e. denoting $L_1^*$
$\Rightarrow$ closed under star-closure
complementation:

$L_1$ is reg. lang.

$\Rightarrow \exists$ DFA $M$ s.t. $L_1 = L(M)$

Construct $M'$ s.t.

final states in $M$ are nonfinal states in $M'$

nonfinal states in $M$ are final states in $M'$

$\Rightarrow$ closed under complementation
intersection:

$L_1$ and $L_2$ are reg. lang.

$\Rightarrow \exists$ DFA $M_1$ and $M_2$ s.t.

$L_1 = L(M_1)$ and $L_2 = L(M_2)$

$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$

$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$

Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$

$Q' = (Q \times P)$

$\delta'$:

$\delta'((q_i, p_j), a) = (q_k, p_l)$ if

$w \in L(M') \iff w \in L_1 \cap L_2$

$\Rightarrow$ closed under intersection
Example:
Regular languages are closed under

reversal \( L^R \)
difference \( L_1 - L_2 \)
right quotient \( L_1 / L_2 \)
homomorphism \( h(L) \)
Right quotient

Def: \( L_1/L_2 = \{ x \mid xy \in L_1 \text{ for some } y \in L_2 \} \)

Example:

\[ L_1 = \{ a^*b^* \cup b^*a^* \} \]
\[ L_2 = \{ b^n \mid n \text{ is even, } n > 0 \} \]
\[ L_1/L_2 = \]
Theorem If $L_1$ and $L_2$ are regular, then $L_1/L_2$ is regular.

Proof (sketch)

$\exists$ DFA $M=(Q,\Sigma,\delta,q_0,F)$ s.t. $L_1 = L(M)$.

Construct DFA $M'=(Q,\Sigma,\delta,q_0,F')$

For each state $i$ do

Make $i$ the start state (representing $L_i'$) if $L_i' \cap L_2 \neq \emptyset$ then put $q_i$ in $F'$ in $M'$

QED.
Homomorphism

Def. Let $\Sigma, \Gamma$ be alphabets. A homomorphism is a function

$$h: \Sigma \rightarrow \Gamma^*$$

Example:

$\Sigma = \{a, b, c\}$, $\Gamma = \{0, 1\}$

- $h(a) = 11$
- $h(b) = 00$
- $h(c) = 0$

$h(bc) = \quad$

$h(ab^*) = \quad$
Questions about regular languages:
L is a regular language.

• Given L, Σ, w ∈ Σ*, is w ∈ L?

• Is L empty?

• Is L infinite?

• Does L₁ = L₂?
Identifying Nonregular Languages

If a language $L$ is finite, is $L$ regular?

If $L$ is infinite, is $L$ regular?

- $L_1 = \{a^n b^m | n > 0, m > 0\}$
- $L_2 = \{a^n b^n | n > 0\}$
Prove that \( L_2 = \{a^n b^n | n > 0\} \) is ?

- Proof:
Pumping Lemma: Let $L$ be an infinite regular language. There exists a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

$$|xy| \leq m$$
$$|y| \geq 1$$
$$xy^iz \in L \text{ for all } i \geq 0$$
To Use the Pumping Lemma to prove $L$ is not regular:

- **Proof by Contradiction.**
  
  Assume $L$ is regular.
  
  $\Rightarrow$ $L$ satisfies the pumping lemma.
  
  Choose a long string $w$ in $L$, $|w| \geq m$.
  
  Show that there is NO division of $w$ into $xyz$ (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $xy^iz \in L \forall i \geq 0$.
  
  The pumping lemma does not hold. Contradiction!
  
  $\Rightarrow$ $L$ is not regular. QED.
Example $L = \{a^n c b^n | n > 0\}$

$L$ is not regular.

• Proof:
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w =$
Example $L = \{a^n b^{n+s} c^s | n, s > 0\}$

$L$ is not regular.

- **Proof:**
  
  Assume $L$ is regular.
  
  $\Rightarrow$ the pumping lemma holds.
  
  Choose $w =$
  
  So the partition is:
Example $\Sigma = \{a, b\}$,
$L = \{w \in \Sigma^* \mid n_a(w) > n_b(w)\}$

$L$ is not regular.

- Proof:
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w =$
  So the partition is:
Example $L = \{a^3 b^n c^{n-3} | n > 3\}$

$L$ is not regular.
To Use Closure Properties to prove $L$ is not regular:

**Proof Outline:**

Assume $L$ is regular.

Apply closure properties to $L$ and other regular languages, constructing $L'$ that you know is not regular.

closure properties $\Rightarrow$ $L'$ is regular.

Contradiction!

$L$ is not regular. QED.
Example \( L = \{ a^3 b^n c^{n-3} | n > 3 \} \)

\( L \) is not regular.

- Proof: (proof by contradiction)
  
  Assume \( L \) is regular.
  
  Define a homomorphism \( h : \Sigma \rightarrow \Sigma^* \)
  
  \[
  h(a) = a \quad h(b) = a \quad h(c) = b
  \]
  
  \( h(L) = \)
Example $L = \{a^n b^m a^m | m \geq 0, n \geq 0\}$

$L$ is not regular.

- **Proof:** (proof by contradiction)
  Assume $L$ is regular.
Example: $L_1 = \{a^n b^n a^n | n > 0\}$

$L_1$ is not regular.