1. (15 points) In how many ways can three distinct numbers be chosen from the set \( \{1, 2, \ldots, 100\} \) such that their sum is even?

2. In the class, we have shown the Pascal’s Identity,

\[
\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.
\]

(a) (15 points) Use it and algebraic manipulation to show

\[
\binom{2m}{m+1} + \binom{2m}{m} = \binom{2m+2}{m+1} \cdot \frac{1}{2}.
\]

(b) (15 points) Let \( n \) and \( k \) be integers with \( 1 \leq k \leq n \). Show that

\[
\sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1} = \binom{2n+2}{n+1} \cdot \frac{1}{2} - \binom{2n}{n}.
\]

3. (15 points) Find the coefficient of \( x^7 \) in the expansion

\[(1 + 2x)^4 (1 - 2x)^6\]

by using the binomial theorem and the product rule. Solution by expanding the expression has no credit.

4. (a) (15 points) Find a recurrence relation for the number of ways to climb \( n \) stairs if the person climbing the stairs can take one or two stairs at a time.

(b) (10 points) What are the initial conditions?

(c) (15 points) Solve the recurrence relation?

5. (20 points) (Bonus) Give a combinatorial proof that

\[
\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}.
\]

Note: this is \#30 of §5.4, see also \#29 for hints.