Lecture 22, Matrix

Matrix is an rectangular array of numbers. An $m \times n$ matrix has $m$ rows and $n$ columns. If $m = n$, we call it a square matrix. Two matrices are equal if they have the same number of rows and columns, and the corresponding entries in every position are equal.

We usually represent a matrix as follows:

$$A = [a_{ij}] = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & 0
\end{bmatrix}$$

Matrix arithmetic: Let $A$ and $B$ be both $m \times n$ matrices,

1. $A + B = [a_{ij} + b_{ij}]$
2. $kA = [ka_{ij}]$, $k$ constant

The product of matrices $A$ and $B$ is defined if $A$ is $m \times k$ and $B$ is $k \times n$ matrices. In other words, $AB$ is defined if the column number of $A$ is the same as the row number of $B$. The dimensionality of $AB$ is $m \times n$. The entries in $AB$ are defined by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}.$$ 

The task is to take the $i$th row of $A$, and the $j$th column of $B$, multiply the corresponding element, and sum the result. For square matrices, we can then define the power of the matrices.

Example 1.

$$\begin{bmatrix}1 & 0 & 4 \\2 & 1 & 1 \\3 & 1 & 0 \\0 & 2 & 2\end{bmatrix} \cdot \begin{bmatrix}2 & 4 \\1 & 1 \\3 & 1\end{bmatrix} = \begin{bmatrix}14 & 4 \\8 & 9 \\7 & 13 \\8 & 2\end{bmatrix}$$

Notice matrix product is not commutative in general.

Example 2. Let

$$A = \begin{bmatrix}1 & 1 \\2 & 9\end{bmatrix}, \quad B = \begin{bmatrix}2 & 1 \\2 & 1\end{bmatrix}$$

Then,

$$AB = \begin{bmatrix}3 & 2 \\5 & 1\end{bmatrix}, \quad BA = \begin{bmatrix}4 & 3 \\3 & 3\end{bmatrix}$$

There is a special square matrix called identity matrix:

$$I_n = \begin{bmatrix}1 & 0 & 0 & 0 \\0 & 1 & 0 & 0 \\\vdots & \vdots & \vdots & \vdots \\0 & 0 & 0 & 1\end{bmatrix}$$

Entries are all zero other than the ones on the diagonal.
Given any $m \times n$ matrix $A$, we have

$$AI_n = I_mA = A$$

There is an counterpart of the division operation in matrix operations (requiring square matrices). It is called the inverse of matrix, denoted by $A^{-1}$. An $n \times n$ matrix $A$ has an inverse if there exists a matrix $B$ such that

$$AB = BA = I_n.$$ 

$B$ is the inverse matrix of $A$. A matrix having inverse is called a nonsingular matrix, otherwise it is called singular. Not every matrix has inverse.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

The transpose of an $m \times n$ matrix $A$ is an $n \times m$ matrix $B$, such that $a_{ij} = b_{ji}$. In other words, the $i$th row, $j$th column entry of $A$ appears at $j$th row, and $i$th column of $B$. The transpose of $A$ is usually denoted by $A'$ or $A^T$. A square matrix $A$ is called symmetric if $A = A^T$. For example, the adjacency matrix of an undirected graph is symmetric.

Path is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph. The length of the path equals to the number of edges traveled.

**Theorem 1.** Let $G$ be a graph with adjacency matrix $A$ with respect to the ordering $v_1, v_2, \ldots, v_n$. The number of different paths of length $r$ from $v_i$ to $v_j$, where $r$ is a positive integer, equals to the $(i, j)$th entry of $A^r$.

**Proof.** We use mathematical induction.

First, the number of paths from $v_i$ to $v_j$ of length 1 is the $(i, j)$ entry of $A$, because this entry is the number of edges from $v_i$ to $v_j$.

Assume that the $(i, j)$th entry of $A^r$ is the number of different paths of length $r$ from $v_i$ to $v_j$. This is the induction hypothesis. Because $A^{r+1} = A^rA$, the $(i, j)$th entry of $A^{r+1}$ equals

$$b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{in}a_{nj},$$

where $b_{ik}$ is the $(i, k)$th entry of $A^r$. By induction hypothesis, $b_{ik}$ is the number of paths of length $r$ from $v_i$ to $v_k$.

A path of length $r+1$ from $v_i$ to $v_j$ is made up of a path of length $r$ from $v_i$ to some intermediate vertex $v_k$ and an edge from $v_k$ to $v_j$. By the product rule, the number of such paths is the product of the number of paths of length $r$ from $v_i$ to $v_k$, namely, $b_{ik}$, and the number of edges from $v_k$ to $v_j$, namely, $a_{kj}$. When these products are added for all possible intermediate vertices $v_k$, the desired result follows by the sum rule for counting. □

$A^r$ can be used to determine the length of the shortest path between two vertices and can also be used to determine if two vertices are connected in the graph.