1. (15 points) Compute
\[
\begin{pmatrix}
1 & 3 & 0 \\
2 & 5 & 7 \\
3 & 6 & 0 \\
\end{pmatrix}
\cdot
\begin{pmatrix}
3 & 4 & 4 & 2 \\
5 & 5 & 6 & 7 \\
6 & 8 & 9 & 10 \\
\end{pmatrix}
\]

**Solution:**
\[
\begin{pmatrix}
18 & 19 & 22 & 23 \\
73 & 89 & 101 & 109 \\
39 & 42 & 48 & 48 \\
\end{pmatrix}
\]

2. (15 points) Show that isomorphism of simple graphs is an equivalent relation.

**Solution:** We need to verify the reflexivity, symmetry, and transitivity.

Reflexivity: use identity function as the isomorphism function.

Symmetry: If \( f \) is the isomorphism between \( G_1 \) and \( G_2 \). Then the inverse of \( f \) is the isomorphism between \( G_2 \) and \( G_1 \).

Transitivity: If \( f \) is the isomorphism between \( G_1 \) and \( G_2 \), and \( g \) is the isomorphism between \( G_2 \) and \( G_3 \), then \( gf \) is the isomorphism from \( G_1 \) to \( G_3 \).

Use the definition of isomorphism to check. If \( a \) and \( b \) are connected in \( G_1 \) then \( f(a) \) and \( f(b) \) must be connected in \( G_2 \), and then \( g(f(a)) \) and \( g(f(b)) \) must be connected in \( G_3 \). This shows \( G_1 \) and \( G_3 \) are isomorphic.

3. (15 points) Show that the property that a graph is bipartite is an isomorphic invariant.

**Solution:** If \( G \) and \( H \) are isomorphic and \( G \) is a bipartite graph, we show \( H \) is also a bipartite graph.

By definition, the vertex set of \( G \) can be divided into two disjoint subsets \( V_1 \) and \( V_2 \) such that each edge in \( G \) has an endpoint in \( V_1 \) and the other one in \( V_2 \).

Let \( f \) be the isomorphism function between \( G \) and \( H \). Then let \( W_1 = f(V_1) \) and \( W_2 = f(V_2) \). As \( f \) is a bijective function, \( W_1 \) and \( W_2 \) are disjoint since \( V_1 \) and \( V_2 \) are. Also, \( W_1 \cup W_2 \) is the vertex set of \( H \).

We only need to verify that every edge in \( H \) has an endpoint in \( W_1 \) and the other one in \( W_2 \). As \( G \) and \( H \) are isomorphic, then for every distinct vertices \( a \) and \( b \) in \( G \), they are adjacent if and only if \( f(a) \) and \( f(b) \) are adjacent. Therefore, for any edge \( e = \{a, b\} \) in \( G \), we can find a corresponding one \( e' = \{f(a), f(b)\} \) in \( H \). As \( G \)
is bipartite, one of the vertices is in $V_1$ and the other one is in $V_2$, meaning one of $f(a)$ and $f(b)$ is in $W_1$ and the other one is in $W_2$. Therefore, $H$ is bipartite.

4. (15 points) Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.

**Solution:** $G$ and $H$ both have 8 vertices, 12 edges, and each vertex has degree 3. There exists circuit of length 3 to 8 in each graph. We will use a length 8 circuit to find the isomorphism.

Since the vertices have the same degree, the construction needs a little bit more observation. We realize that there exist two length 3 circuits in each graph which are connected by a single edge. For $G$, it is $u_1, u_2, u_3$ and $u_8, u_7, u_6$, the two are connected by edge $\{u_1,u_8\}$. For $H$, it is $v_1, v_2, v_3$ and $v_5, v_6, v_7$, the two are connected by edge $\{v_2,v_6\}$.

We use this particular information, build a length 8 circuit in $G$ as $u_3, u_2, u_1, u_8, u_7, u_6, u_5, u_4$.

The two length 3 circuits are traversed one after the other. This leads to a mapping $v_3, v_1, v_2, v_6, v_5, v_7, v_8, v_4$.

With this particular order, we write down the adjacency matrices for each graph, and find they are the same as follows.

\[
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}
\]
5. (20 points) Show that if a simple graph $G$ has $k$ connected components and these components have $n_1, n_2, \ldots, n_k$ vertices, respectively, then the number of edges of $G$ does not exceed

$$\sum_{i=1}^{k} C(n_i, 2)$$

Solution: Each connect component with $n_i$ vertices can have at most $C(n_i, 2)$ edges. That corresponds to the case every distinct vertices are connected with an edge. Since $G$ is a simple graph, there can be at most one edge between any distinct vertices.

6. (15 points) (Bonus) Use previous result to show that a simple graph with $n$ vertices and $k$ connected components has at most $\frac{(n-k)(n-k+1)}{2}$ edges.

Hint: First, show that

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - (k - 1)(2n - k),$$

where $n_i$ is the number of vertices in the $i$th connected component. To show that, consider

$$\sum_{i=1}^{k} (n_i - 1) = n - k$$

Solution: Square on both sides of $\sum_{i=1}^{k} (n_i - 1) = n - k$, we then have

$$\sum_{i=1}^{k} (n_i - 1)^2 + \text{cross terms} = n^2 - 2nk + k^2$$

The left hand side can be simplified to

$$\sum_{i=1}^{k} (n_i - 1)^2 = \sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} 2n_i + k = \sum_{i=1}^{k} n_i^2 - 2n + k$$

Then, we have

$$\sum_{i=1}^{k} n_i^2 \leq n^2 - 2nk + k^2 + 2n - k = n^2 - (k - 1)(2n - k)$$

because all the cross terms are non-negative.
Then, use the result of # 5, we have the number of edges is at most

\[
\sum_{i=1}^{k} C(n_i, 2) = \sum_{i=1}^{k} (n_i - 1)n_i/2 = \frac{1}{2} \sum_{i=1}^{k} n_i^2 - \frac{n}{2}
\]

\[
\leq \frac{n^2 - (k - 1)(2n - k) - n}{2} = \frac{n^2 - 2nk + k^2 + n - k}{2} = \frac{(n - k)(n - k + 1)}{2}
\]

7. (20 points) Use previous result to show that a simple graph with \( n \) vertices is connected if it has more than \( \frac{(n-1)(n-2)}{2} \) edges.

**Solution:** Notice that the value of \( (n - k)(n - k + 1)/2 \) decreases when \( k \) becomes larger.

If a simple graph with \( n \) vertices is not connected, it will contain at least 2 connected components. Therefore, the value of \( k \) in previous problem is \( k \geq 2 \).

Then, there are at most \( (n - 2)(n - 2 + 1)/2 \) edges in the graph, which contradicts to the condition that the graph has more than \( (n - 1)(n - 2)/2 \) edges.

Therefore, the graph is connected.