CompSci 102
Discrete Math for Computer Science

Announcements
- These slides on Chapt 2.3-2.4
- Read for next time Chap 13.3
- Homework 2 due on Thursday
- Recitation on Friday this week

January 31, 2012
Prof. Rodger

Slides modified from Rosen

Functions
- Section 2.3

Definition: Let $A$ and $B$ be nonempty sets. A function $f$ from $A$ to $B$, denoted $f: A \to B$ is an assignment of each element of $A$ to exactly one element of $B$. We write $f(a) = b$ if $b$ is the unique element of $B$ assigned by the function $f$ to the element $a$ of $A$.

- Functions are sometimes called mappings or transformations.

Students
- Carlota Rodriguez
- Sandeep Patel
- Jalen Williams
- Kathy Scott

Grades
- A
- B
- C
- D
- F
Functions

• A function \( f: A \rightarrow B \) can also be defined as a subset of \( A \times B \) (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.

• Specifically, a function \( f \) from \( A \) to \( B \) contains one, and only one ordered pair \( (a, b) \) for every element \( a \in A \).

\[ \forall x \exists y \left[ x \in A \rightarrow (x, y) \in f \right] \text{ and } \forall x, y_1, y_2 \left[ (x, y_1) \in f \land (x, y_2) \in f \rightarrow y_1 = y_2 \right]. \]

Representing Functions

• Functions may be specified in different ways:
  – An explicit statement of the assignment. Students and grades example.
  – A formula.
    \( f(x) = x + 1 \)
  – A computer program.
    • A Java program that when given an integer \( n \), produces the \( n \)th Fibonacci Number (covered in the next section and also in Chapter 5).

Questions

\( f(a) = ? \)

The image of \( d \) is ?

The domain of \( f \) is ?

The codomain of \( f \) is ?

The preimage of \( y \) is ?

\( f(A) = ? \)

The preimage(s) of \( z \) is (are) ?
Questions

\( f(a) = ? \quad z \)

The image of d is ? \( z \)

The domain of \( f \) is ? \( A \)

The codomain of \( f \) is ? \( B \)

The preimage of y is ? \( b \)

\( f(A) = ? \quad \{y,z\} \)

The preimage(s) of z is (are) ? \( \{a,c,d\} \)

Question on Functions and Sets

- If \( f : A \to B \) and \( S \) is a subset of \( A \), then
  \[ f(S) = \{ f(s) \mid s \in S \} \]

\( f\{a,b,c\} \) is ? \( \{y,z\} \)

\( f\{c,d\} \) is ? \( \{z\} \)

Injections

**Definition:** A function \( f \) is said to be **one-to-one**, or **injective**, if and only if \( f(a) = f(b) \) implies that \( a = b \) for all \( a \) and \( b \) in the domain of \( f \). A function is said to be an **injection** if it is one-to-one.
Surjections

Definition: A function \( f \) from \( A \) to \( B \) is called onto or surjective, if and only if for every element \( a \in A \) there is an element \( b \in B \) with \( f(a) = b \). A function \( f \) is called a surjection if it is onto.

\[ A \quad B \]
\[ \begin{array}{c}
\text{a} \\
\text{b} \\
\text{c} \\
\text{d}
\end{array} \quad \begin{array}{c}
x \\
y \\
z
\end{array} \]

Showing that \( f \) is one-to-one or onto

Suppose that \( f : A \to B \).

To show that \( f \) is injective Show that if \( f(x) = f(y) \) for arbitrary \( x, y \in A \) with \( x \neq y \), then \( x = y \).

To show that \( f \) is not injective Find particular elements \( x, y \in A \) such that \( x \neq y \) and \( f(x) = f(y) \).

To show that \( f \) is surjective Consider an arbitrary element \( y \in B \) and find an element \( x \in A \) such that \( f(x) = y \).

To show that \( f \) is not surjective Find a particular \( y \in B \) such that \( f(x) \neq y \) for all \( x \in A \).

Bijections

Definition: A function \( f \) is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto (surjective and injective).

\[ A \quad B \]
\[ \begin{array}{c}
a \\
b \\
c \\
d
\end{array} \quad \begin{array}{c}
x \\
y \\
z \\
w
\end{array} \]

Showing that \( f \) is one-to-one or onto

Example 1: Let \( f \) be the function from \( \{a,b,c,d\} \) to \( \{1,2,3\} \) defined by \( f(a) = 3, f(b) = 2, f(c) = 1, \) and \( f(d) = 3 \). Is \( f \) an onto function?

Example 2: Is the function \( f(x) = x^2 \) from the set of integers onto?
Showing that $f$ is one-to-one or onto

**Example 1:** Let $f$ be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is $f$ an onto function?

**Solution:** Yes, $f$ is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to $\{1,2,3,4\}$, $f$ would not be onto.

**Example 2:** Is the function $f(x) = x^2$ from the set of integers onto?

**Solution:** No, $f$ is not onto because there is no integer $x$ with $x^2 = -1$, for example.

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**Inverse Functions**

**Definition:** Let $f$ be a bijection from $A$ to $B$. Then the *inverse* of $f$, denoted $f^{-1}$, is the function from $B$ to $A$ defined as

$$f^{-1}(y) = x \text{ iff } f(x) = y$$

No inverse exists unless $f$ is a bijection. Why?

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**Questions**

**Example 1:** Let $f$ be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is $f$ invertible and if so what is its inverse?
Questions

Example 1: Let \( f \) be the function from \( \{a,b,c\} \) to \( \{1,2,3\} \) such that \( f(a) = 2 \), \( f(b) = 3 \), and \( f(c) = 1 \). Is \( f \) invertible and if so what is its inverse?

Solution: The function \( f \) is invertible because it is a one-to-one correspondence. The inverse function \( f^{-1} \) reverses the correspondence given by \( f \), so \( f^{-1}(1) = c \), \( f^{-1}(2) = a \), and \( f^{-1}(3) = b \).

Questions

Example 2: Let \( f: \mathbb{Z} \rightarrow \mathbb{Z} \) be such that \( f(x) = x + 1 \). Is \( f \) invertible, and if so, what is its inverse?

Solution: The function \( f \) is invertible because it is a one-to-one correspondence. The inverse function \( f^{-1} \) reverses the correspondence so \( f^{-1}(y) = y - 1 \).

Questions

Example 3: Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) be such that \( f(x) = x^2 \). Is \( f \) invertible, and if so, what is its inverse?
Questions

Example 3: Let $f: \mathbb{R} \to \mathbb{R}$ be such that $f(x) = x^2$. Is $f$ invertible, and if so, what is its inverse?

Solution: The function $f$ is not invertible because it is not one-to-one.

Composition

- **Definition**: Let $f: B \to C$, $g: A \to B$. The composition of $f$ with $g$, denoted $f \circ g$, is the function from $A$ to $C$ defined by $f \circ g(x) = f(g(x))$.

Example 1: If $f(x) = x^2$ and $g(x) = 2x + 1$ then $f(g(x)) =$ and $g(f(x)) =$
Composition

**Example 1:** If \( f(x) = x^2 \) and \( g(x) = 2x + 1 \) then
\[
f(g(x)) = (2x + 1)^2
\]
and
\[
g(f(x)) = 2x^2 + 1
\]

Composition Questions

**Example 2:** Let \( g \) be the function from the set \( \{a, b, c\} \) to itself such that \( g(a) = b, g(b) = c, \) and \( g(c) = a \). Let \( f \) be the function from the set \( \{a, b, c\} \) to the set \( \{1, 2, 3\} \) such that \( f(a) = 3, f(b) = 2, \) and \( f(c) = 1 \).
What is the composition of \( f \) and \( g \), and what is the composition of \( g \) and \( f \)?

**Solution:**

The composition \( f \circ g \) is defined by

\[
\begin{align*}
(f \circ g)(a) &= f(g(a)) = f(b) = 2. \\
(f \circ g)(b) &= f(g(b)) = f(c) = 1. \\
(f \circ g)(c) &= f(g(c)) = f(a) = 3.
\end{align*}
\]

Note that \( g \circ f \) is not defined, because the range of \( f \) is not a subset of the domain of \( g \).
Composition Questions

Example 3: Let $f$ and $g$ be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

What is the composition of $f$ and $g$, and also the composition of $g$ and $f$?

Solution:

$$f \circ g(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

$$g \circ f(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$$

Some Important Functions

- The floor function, denoted
  $$f(x) = \lfloor x \rfloor$$
  is the largest integer less than or equal to $x$.

- The ceiling function, denoted
  $$f(x) = \lceil x \rceil$$
  is the smallest integer greater than or equal to $x$.

Example:

$$\lfloor 3.5 \rfloor = 4 \quad \lfloor -1.5 \rfloor = -1$$

$$\lceil 3.5 \rceil = 3 \quad \lceil -1.5 \rceil = -1$$

Graphs of Functions

- Let $f$ be a function from the set $A$ to the set $B$. The graph of the function $f$ is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.

Some Important Functions

- The floor function, denoted
  $$f(x) = \lfloor x \rfloor$$
  is the largest integer less than or equal to $x$.

- The ceiling function, denoted
  $$f(x) = \lceil x \rceil$$
  is the smallest integer greater than or equal to $x$.

Example:

$$\lfloor 3.5 \rfloor = 4 \quad \lfloor -1.5 \rfloor = -1$$

$$\lceil 3.5 \rceil = 3 \quad \lceil -1.5 \rceil = -2$$
Floor and Ceiling Functions

<table>
<thead>
<tr>
<th>TABLE 1</th>
<th>Useful Properties of the Floor and Ceiling Functions.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n is an integer, x is a real number)</td>
<td></td>
</tr>
<tr>
<td>(1a) ([x] = n) if and only if (n \leq x &lt; n + 1)</td>
<td></td>
</tr>
<tr>
<td>(1b) ([x] = n) if and only if (n - 1 &lt; x \leq n)</td>
<td></td>
</tr>
<tr>
<td>(1c) ([x] = n) if and only if (x - 1 &lt; n \leq x)</td>
<td></td>
</tr>
<tr>
<td>(1d) ([x] = n) if and only if (x \leq n &lt; x + 1)</td>
<td></td>
</tr>
<tr>
<td>(2) (x - 1 &lt; [x] \leq x &lt; x + 1)</td>
<td></td>
</tr>
<tr>
<td>(3a) ([-x] = -[x])</td>
<td></td>
</tr>
<tr>
<td>(3b) ([-x] = -[x])</td>
<td></td>
</tr>
<tr>
<td>(4a) ([x + n] = [x] + n)</td>
<td></td>
</tr>
<tr>
<td>(4b) ([x + n] = [x] + n)</td>
<td></td>
</tr>
</tbody>
</table>

Proving Properties of Functions

Example: Prove that \(x\) is a real number, then
\([2x] = [x] + [x + 1/2]\)

Solution: Let \(x = n + \varepsilon\), where \(n\) is an integer and \(0 \leq \varepsilon < 1\).

Case 1: \(\varepsilon < \frac{1}{2}\)
- \(2x = 2n + 2\varepsilon\) and \([2x] = 2n\), since \(0 \leq 2\varepsilon < 1\).
- \([x + 1/2] = n\), since \(x + \varepsilon = n + (1/2 + \varepsilon)\) and \(0 \leq \varepsilon < 1\).
- Hence, \([2x] = 2n\) and \([x] + [x + 1/2] = n + n = 2n\).

Case 2: \(\varepsilon \geq \frac{1}{2}\)
- \(2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)\) and \([2x] = 2n + 1\), since \(0 \leq 2\varepsilon - 1 < 1\).
- \([x + 1/2] = [n + (1/2 + \varepsilon)] = [n + 1 + (\varepsilon - 1/2)] = n + 1\) since \(0 \leq \varepsilon - 1/2 < 1\).
- Hence, \([2x] = 2n + 1\) and \([x] + [x + 1/2] = n + (n + 1) = 2n + 1\).

Factorial Function

Definition: \(f: \mathbb{N} \to \mathbb{Z}^+\), denoted by \(f(n) = n!\)

is the product of the first \(n\) positive integers when \(n\) is a nonnegative integer.

\(f(n) = 1 \cdot 2 \cdots (n - 1) \cdot n, \quad f(0) = 0! = 1\)

Examples:
- \(f(1) = 1! = 1\)
- \(f(2) = 2! = 1 \cdot 2 = 2\)

Stirling’s Formula:
- \(n! \sim \sqrt{2\pi n}(n/e)^n\)
- \(f(n) \sim g(n) \Rightarrow \lim_{n \to \infty} f(n)/g(n) = 1\)
- \(f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720\)
- \(f(20) = 2,432,902,008,176,640,000\).
Partial Functions

**Definition**: A *partial function* $f$ from a set $A$ to a set $B$ is an assignment to each element $a$ in a subset of $A$, called the *domain of definition* of $f$, of a unique element $b$ in $B$.

- The sets $A$ and $B$ are called the *domain* and *codomain* of $f$, respectively.
- We say that $f$ is *undefined* for elements in $A$ that are not in the domain of definition of $f$.
- When the domain of definition of $f$ equals $A$, we say that $f$ is a *total function*.

**Example**: $f : \mathbb{Z} \rightarrow \mathbb{R}$ where $f(n) = \sqrt{n}$ is a partial function from $\mathbb{Z}$ to $\mathbb{R}$ where the domain of definition is the set of nonnegative integers. Note that $f$ is undefined for negative integers.

Sequences and Summations

- **Section 2.4**

Introduction

- Sequences are ordered lists of elements.
  - 1, 2, 3, 5, 8
  - 1, 3, 9, 27, 81, .......
- Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.
- We will introduce the terminology to represent sequences and sums of the terms in the sequences.

Sequences

**Definition**: A *sequence* is a function from a subset of the integers (usually either the set \{0, 1, 2, 3, 4, ....\} or \{1, 2, 3, 4, ....\}) to a set $S$.

- The notation $a_n$ is used to denote the image of the integer $n$. We can think of $a_n$ as the equivalent of $f(n)$ where $f$ is a function from \{0,1,2,.....\} to $S$. We call $a_n$ a *term* of the sequence.
Sequences

**Example:** Consider the sequence \( \{a_n\} \)
where
\[
a_n = \frac{1}{n} \quad \{a_n\} = \{a_1, a_2, a_3, \ldots\}
\]

\[
1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots
\]

Geometric Progression

**Definition:** A *geometric progression* is a sequence of the form:
\[
a, ar, ar^2, \ldots, ar^n, \ldots
\]
where the *initial term* \( a \) and the *common ratio* \( r \) are real numbers.

**Examples:**
1. Let \( a = 1 \) and \( r = -1 \). Then:
   \[
   \{b_n\} = \{b_0, b_1, b_2, b_3, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}
   \]
2. Let \( a = 2 \) and \( r = 5 \). Then:
   \[
   \{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}
   \]
3. Let \( a = 6 \) and \( r = 1/3 \). Then:
   \[
   \{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \{6, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\}
   \]

Arithmetic Progression

**Definition:** A *arithmetic progression* is a sequence of the form:
\[
\begin{align*}
a, a + d, a + 2d, \ldots, a + nd, \ldots
\end{align*}
\]
where the *initial term* \( a \) and the *common difference* \( d \) are real numbers.

**Examples:**
1. Let \( a = -1 \) and \( d = 4 \):
   \[
   \{e_n\} = \{-1, 3, 7, 11, \ldots\}
   \]
2. Let \( a = 7 \) and \( d = -3 \):
   \[
   \{f_n\} = \{7, 4, 1, -2, \ldots\}
   \]
3. Let \( a = 1 \) and \( d = 2 \):
   \[
   \{g_n\} = \{1, 3, 5, 7, \ldots\}
   \]
**Arithmetic Progression**

**Definition:** A arithmetic progression is a sequence of the form: $a, a + d, a + 2d, \ldots, a + nd, \ldots$

where the initial term $a$ and the common difference $d$ are real numbers.

**Examples:**

1. Let $a = -1$ and $d = 4$:
   \[
   \{s_n\} = \{s_0, s_1, s_2, s_3, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\}
   \]

2. Let $a = 7$ and $d = -3$:
   \[
   \{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}
   \]

3. Let $a = 1$ and $d = 2$:
   \[
   \{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}
   \]

**Recurrence Relations**

**Definition:** A recurrence relation for the sequence \{a_n\} is an equation that expresses $a_n$ in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers $n$ with $n \geq n_0$, where $n_0$ is a nonnegative integer.

- A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.
- The initial conditions for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

**Strings**

**Definition:** A string is a finite sequence of characters from a finite set (an alphabet).

- Sequences of characters or bits are important in computer science.
- The empty string is represented by \emptyset.
- The string $abcde$ has length 5.

**Questions about Recurrence Relations**

**Example 1:** Let \{a_n\} be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, 4, \ldots$ and suppose that $a_0 = 2$. What are $a_1$, $a_2$ and $a_3$? [Here $a_0 = 2$ is the initial condition.]
Questions about Recurrence Relations

**Example 1:** Let \( \{a_n\} \) be a sequence that satisfies the recurrence relation \( a_n = a_{n-1} + 3 \) for \( n = 1, 2, 3, 4, \ldots \) and suppose that \( a_0 = 2 \). What are \( a_1 \), \( a_2 \) and \( a_3 \)? [Here \( a_0 = 2 \) is the initial condition.]

**Solution:** We see from the recurrence relation that
\[
\begin{align*}
a_1 &= a_0 + 3 = 2 + 3 = 5 \\
a_2 &= a_1 + 3 = 5 + 3 = 8 \\
a_3 &= a_2 + 3 = 8 + 3 = 11
\end{align*}
\]

**Example 2:** Let \( \{a_n\} \) be a sequence that satisfies the recurrence relation \( a_n = a_{n-1} - a_{n-2} \) for \( n = 2, 3, 4, \ldots \) and suppose that \( a_0 = 3 \) and \( a_1 = 5 \). What are \( a_2 \) and \( a_3 \)? [Here the initial conditions are \( a_0 = 3 \) and \( a_1 = 5 \).]

**Solution:** We see from the recurrence relation that
\[
\begin{align*}
a_2 &= a_1 - a_0 = 5 - 3 = 2 \\
a_3 &= a_2 - a_1 = 2 - 5 = -3
\end{align*}
\]

Questions about Recurrence Relations

**Example 2:** Let \( \{a_n\} \) be a sequence that satisfies the recurrence relation \( a_n = a_{n-1} - a_{n-2} \) for \( n = 2, 3, 4, \ldots \) and suppose that \( a_0 = 3 \) and \( a_1 = 5 \). What are \( a_2 \) and \( a_3 \)? [Here the initial conditions are \( a_0 = 3 \) and \( a_1 = 5 \).]

**Solution:** We see from the recurrence relation that
\[
\begin{align*}
a_2 &= a_1 - a_0 = 5 - 3 = 2 \\
a_3 &= a_2 - a_1 = 2 - 5 = -3
\end{align*}
\]

Fibonacci Sequence

**Definition:** Define the Fibonacci sequence, \( f_0, f_1, f_2, \ldots \), by:
- Initial Conditions: \( f_0 = 0 \), \( f_1 = 1 \)
- Recurrence Relation: \( f_n = f_{n-1} + f_{n-2} \)

**Example:** Find \( f_2, f_3, f_4, f_5 \) and \( f_6 \).

**Answer:**
\[
f_2 = f_1 + f_0 = 1 + 0 = 1,
\]
\[
f_3 = f_2 + f_1 = 1 + 1 = 2,
\]
\[
f_4 = f_3 + f_2 = 2 + 1 = 3,
\]
\[
f_5 = f_4 + f_3 = 3 + 2 = 5,
\]
\[
f_6 = f_5 + f_4 = 5 + 3 = 8.
\]
Fibonacci Sequence

**Definition:** Define the *Fibonacci sequence* \( f_0, f_1, f_2, \ldots \) by:
- Initial Conditions: \( f_0 = 0, f_1 = 1 \)
- Recurrence Relation: \( f_n = f_{n-1} + f_{n-2} \)

**Example:** Find \( f_2, f_3, f_4, f_5 \) and \( f_6 \).

**Answer:**
- \( f_2 = f_1 + f_0 = 1 + 0 = 1 \),
- \( f_3 = f_2 + f_1 = 1 + 1 = 2 \),
- \( f_4 = f_3 + f_2 = 2 + 1 = 3 \),
- \( f_5 = f_4 + f_3 = 3 + 2 = 5 \),
- \( f_6 = f_5 + f_4 = 5 + 3 = 8 \).

**Solving Recurrence Relations**

Finding a formula for the *n*th term of the sequence generated by a recurrence relation is called *solving the recurrence relation.*

Such a formula is called a *closed formula.*

Various methods for solving recurrence relations will be covered in Chapter 8 where recurrence relations will be studied in greater depth.

Here we illustrate by example the method of iteration in which we need to guess the formula. The guess can be proved correct by the method of induction (Chapter 5).

**Iterative Solution Example**

- **Method 1:** Working upward, forward substitution
  - Let \( \{a_n\} \) be a sequence that satisfies the recurrence relation \( a_n = a_{n-1} + 3 \) for \( n = 2, 3, 4, \ldots \) and suppose that \( a_1 = 2 \)
  - \( a_2 = 2 + 3 \)
  - \( a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2 \)
  - \( a_4 = (2 + 3 \cdot 2) + 3 = 2 + 3 \cdot 3 \)
  - \( a_5 = (2 + 3 \cdot (k - 2)) + 3 = 2 + 3 \cdot (k - 1) \)
  - \( a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1) \)
Iterative Solution Example

- Method 2: Working downward, backward substitution
- Let \( \{a_n\} \) be a sequence that satisfies the recurrence relation
  \[ a_n = a_{n-1} + 3 \text{ for } n = 2, 3, 4, \ldots \text{ and suppose that } a_1 = 2 \]
- \( a_n = a_{n-1} + 3 \)

Iterative Solution Example

- Method 2: Working downward, backward substitution
- Let \( \{a_n\} \) be a sequence that satisfies the recurrence relation
  \[ a_n = a_{n-1} + 3 \text{ for } n = 2, 3, 4, \ldots \text{ and suppose that } a_1 = 2 \]
- \( a_n = a_{n-1} + 3 \)
  \[ = (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \]
  \[ = (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \]
  \[ \cdots \]
  \[ = (a_{n-k} + 3) + 3 \cdot (k - 1) = a_{n-k} + 3 \cdot k \]
  \[ \cdots \]
  \[ = a_2 + 3(n - 2) = (a_1 + 3) + 3(n - 2) = 2 + 3(n - 1) \]
- note: last step \( k = n - 2 \)

Financial Application

Example: Suppose that a person deposits $10,000.00 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Let \( P_n \) denote the amount in the account after 30 years. \( P_n \) satisfies the following recurrence relation:

\[ P_n = P_{n-1} + 0.11P_{n-1} = (1.11) P_{n-1} \]

with the initial condition \( P_0 = 10,000 \)

Continued on next slide
Financial Application

\[ P_n = P_{n-1} + 0.11P_{n-1} = (1.11)P_{n-1} \]
with the initial condition \( P_0 = 10,000 \)

**Solution:** Forward Substitution
\[ P_1 = (1.11)P_0 \]

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**Solution:** Forward Substitution
\[ P_1 = (1.11)P_0 \]
\[ P_2 = (1.11)P_1 = (1.11)^2P_0 \]
\[ P_3 = (1.11)P_2 = (1.11)^3P_0 \]
\[ \vdots \]
\[ P_k = (1.11)P_{k-1} = (1.11)^kP_0 \]
\[ P_n = (1.11)P_{n-1} = (1.11)^nP_0 = (1.11)^n \times 10,000 \]
\[ P_{30} = (1.11)^{30} \times 10,000 = \$228,992.97 \]

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Special Integer Sequences

Given a few terms of a sequence, try to identify the sequence. Conjecture a formula, recurrence relation, or some other rule.

Some questions to ask?

- Are there repeated terms of the same value?
- Can you obtain a term from the previous term by adding an amount or multiplying by an amount?
- Can you obtain a term by combining the previous terms in some way?
- Are they cycles among the terms?
- Do the terms match those of a well known sequence?

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Questions on Special Integer Sequences

**Example 1:** Find formulae for the sequences with the following first five terms: 1, \( \frac{1}{2} \), \( \frac{1}{4} \), 1/8, 1/16

**Example 2:** Consider 1, 3, 5, 7, 9

**Example 3:** 1, -1, 1, -1, 1
Questions on Special Integer Sequences

Example 1: Find formulae for the sequences with the following first five terms: 1, 1/2, 1/4, 1/8, 1/16
Solution: Note that the denominators are powers of 2. The sequence with \(a_n = 1/2^n\) is a possible match. This is a geometric progression with \(a = 1\) and \(r = 1/2\).

Example 2: Consider 1,3,5,7,9
Solution: Note that each term is obtained by adding 2 to the previous term. A possible formula is \(a_n = 2n + 1\). This is an arithmetic progression with \(a = 1\) and \(d = 2\).

Example 3: 1, -1, 1, -1,1
Solution: The terms alternate between 1 and -1. A possible sequence is \(a_n = (-1)^n\). This is a geometric progression with \(a = 1\) and \(r = -1\).

Useful Sequences

<table>
<thead>
<tr>
<th>nth Term</th>
<th>First 10 Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n^2)</td>
<td>1, 4, 9, 16, 25, 36, 49, 64, 81, 100, …</td>
</tr>
<tr>
<td>(n^3)</td>
<td>1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, …</td>
</tr>
<tr>
<td>(n^4)</td>
<td>1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, …</td>
</tr>
<tr>
<td>(2^n)</td>
<td>2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, …</td>
</tr>
<tr>
<td>(3^n)</td>
<td>3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, …</td>
</tr>
<tr>
<td>(n!)</td>
<td>1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, …</td>
</tr>
<tr>
<td>(f_n)</td>
<td>1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …</td>
</tr>
</tbody>
</table>

Guessing Sequences

Example: Conjecture a simple formula for \(a_n\) if the first 10 terms of the sequence \(\{a_n\}\) are 1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047.

Solution: Note the ratio of each term to the previous approximates 3. So now compare with the sequence \(3^n\). We notice that the \(n\)th term is 2 less than the corresponding power of 3. So a good conjecture is that \(a_n = 3^n - 2\).
Summations

Sum of the terms \( a_m, a_{m+1}, \ldots, a_n \) from the sequence \( \{a_n\} \)

The notation:

\[
\sum_{j=m}^{n} a_j \quad \sum_{j=m}^{n} a_j \quad \sum_{m \leq j \leq n} a_j
\]

represents \( a_m + a_{m+1} + \cdots + a_n \)

The variable \( j \) is called the index of summation. It runs through all the integers starting with its lower limit \( m \) and ending with its upper limit \( n \).

Summations

- More generally for a set \( S \):
  \[
  \sum_{j \in S} a_j
  \]
  \[
  r^0 + r^1 + r^2 + r^3 + \ldots + r^n = \sum_{j=0}^{n} r^j
  \]

- Examples:
  \[
  1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{i=1}^{\infty} \frac{1}{i}
  \]
  If \( S = \{2, 5, 7, 10\} \) then \( \sum_{j \in S} a_j = a_2 + a_5 + a_7 + a_{10} \)

Product Notation

Product of the terms \( a_m, a_{m+1}, \ldots, a_n \) from the sequence \( \{a_n\} \)

The notation:

\[
\prod_{j=m}^{n} a_j \quad \prod_{j=m}^{n} a_j \quad \prod_{m \leq j \leq n} a_j
\]

represents \( a_m \times a_{m+1} \times \cdots \times a_n \)

Example

- Find a formula for \( \sum_{k=1}^{n} 2k - 1 \)
Example

• Find a formula for \( \sum_{k=1}^{n} 2k - 1 \)

\[
\begin{align*}
  a_1 &= 1 \\
  a_2 &= a_1 + 3 \\
  a_3 &= a_2 + 5 \\
  \quad \ldots \\
  a_k &= a_{k-1} + (2k - 1) \\
  \quad \ldots \\
  a_n &= a_{n-1} + (2n - 1)
\end{align*}
\]

Example solve (cont)

• \( a_n = a_{n-1} + 2n - 1 \)

\[
\begin{align*}
  &= [a_{n-2} + 2(n-1) - 1] + 2n - 1 \\
  &= a_{n-2} + 4n - 4 \\
  &= [a_{n-3} + 2(n-2) - 1] + 4n - 4 \\
  &= a_{n-3} + 6n - 9 \\
  \quad \ldots \\
  &= a_{n-k} + 2kn - k^2 \\
  \quad \ldots \\
  &= a_1 + 2(n-1)n - (n-1)^2 = n^2
\end{align*}
\]

Some Useful Summation Formulae

<table>
<thead>
<tr>
<th>Sum</th>
<th>Closed Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \sum_{k=0}^{n} \alpha^k ] (( \alpha \neq 0 ))</td>
<td>[ \frac{\alpha^{n+1} - \alpha}{\alpha - 1} ]</td>
</tr>
<tr>
<td>[ \sum_{k=1}^{n} k ]</td>
<td>[ \frac{n(n+1)}{2} ]</td>
</tr>
<tr>
<td>[ \sum_{k=1}^{n} k^2 ]</td>
<td>[ \frac{n(n+1)(2n+1)}{6} ]</td>
</tr>
<tr>
<td>[ \sum_{k=1}^{n} k^3 ]</td>
<td>[ \frac{n^2(n+1)^2}{4} ]</td>
</tr>
<tr>
<td>[ \sum_{k=0}^{\infty} x^k,</td>
<td>x</td>
</tr>
<tr>
<td>[ \sum_{k=1}^{\infty} \alpha^k,</td>
<td>\alpha</td>
</tr>
</tbody>
</table>

Geometric Series: We just proved this.
Later we will prove some of these by induction.
Proof in text (requires calculus)