Section: Decidability

Computability A function \( f \) with domain \( D \) is *computable* if there exists some TM \( M \) such that \( M \) computes \( f \) for all values in its domain.

Decidability A problem is *decidable* if there exists a TM that can answer yes or no to every statement in the domain of the problem.
The Halting Problem

Domain: set of all TMs and all strings $w$.

Question: Given coding of $M$ and $w$, does $M$ halt on $w$?
Theorem The halting problem is undecidable.

Proof: (by contradiction)

• Assume there is a TM H (or algorithm) that solves this problem.
  TM H has 2 final states, \( q_y \) represents yes and \( q_n \) represents no.

\[
H(w_M, w) = \begin{cases} 
\text{halts } q_y \text{ if } M \text{ halts on } w \\
\text{halts } q_n \text{ if } M \text{ doesn’t halt on } w 
\end{cases}
\]

TM H always halts in a final state.
Construct TM $H'$ from $H$

$$H'(w_M, w) = \begin{cases} \text{halts} & \text{if } M \text{ not halt on } w \\
\text{not halt} & \text{if } M \text{ halts on } w \end{cases}$$

Construct TM $\hat{H}$ from $H'$

$$\hat{H}(w_M) = \begin{cases} \text{halts} & \text{if } M \text{ not halt on } w_M \\
\text{not halt} & \text{if } M \text{ halts on } w_M \end{cases}$$

Note that $\hat{H}$ is a TM.

There is some encoding of it, say $\hat{w}_{\hat{H}}$.

What happens if we run $\hat{H}$ with input $\hat{w}_{\hat{H}}$?
Theorem If the halting problem were decidable, then every recursively enumerable language would be recursive. Thus, the halting problem is undecidable.

- Proof: Let $L$ be an RE language over $\Sigma$.
Let $M$ be the TM such that $L = L(M)$.
Let $H$ be the TM that solves the halting problem.
A problem A is reduced to problem B if the decidability of B follows from the decidability of A. Then if we know B is undecidable, then A must be undecidable.
State-entry problem

Given TM $M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$, state $q \in Q$, and string $w \in \Sigma^*$, is state $q$ ever entered when $M$ is applied to $w$?

This is an undecidable problem!

Proof:

TM $E$ solves state-entry problem

$$E'(w_M, w) = \begin{cases} 
M \text{ halts on } w & \text{if } \text{?} \\
M \text{ doesn't halt on } w & \text{if } \text{?}
\end{cases}$$