Ch. 7 - Pushdown Automata

A DFA = (Q, Σ, δ, q₀, F)

Modify DFA by adding a stack. New machine is called Pushdown Automata (PDA).

Definition: Nondeterministic PDA (NPDA) is defined by

\[ M = (Q, \Sigma, \Gamma, \delta, q₀, z, F) \]
where

- $Q$ is finite set of states
- $\Sigma$ is tape (input) alphabet
- $\Gamma$ is stack alphabet
- $q_0$ is initial state
- $z$ - start stack symbol, (bottom of stack marker), $z \in \Gamma$
- $F \subseteq Q$ is set of final states.

$\delta : Q \times (\Sigma \cup \{\lambda\}) \times \Gamma \rightarrow \text{finite subsets of } Q \times \Gamma^*$

**Example of transitions**

$\delta(q_1, a, b) = \{(q_3, b), (q_4, ab), (q_6, \lambda)\}$

Meaning: If in state $q_1$ with “$a$” the current tape symbol and “$b$” the symbol on top of the stack, then pop “$b$”, and either

- move to $q_3$ and push “$b$” on stack
- move to $q_4$ and push “$ab$” on stack (“$a$” on top)
- move to $q_6$

Transitions can be represented using a transition diagram.

The diagram for the above transitions is:

Each arc is labeled by a triple: $x, y, z$ where $x$ is the current input symbol, $y$ is the top of stack symbol which is popped from the stack, and $z$ is a string that is pushed onto the stack.

**Instantaneous Description:**

$$(q, w, u)$$

Notation to describe the current state of the machine ($q$), unread portion of the input string ($w$), and the current contents of the stack ($u$).
**Description of a Move:**

\[(q_1, aw, bx) \vdash (q_2, w, yx)\]

**iff**

**Definition** Let \(M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)\) be a NPDA. \(L(M) = \{ w \in \Sigma^* \mid (q_0, w, z)^* \vdash (p, \lambda, u), p \in F, u \in \Gamma^* \}\). The NPDA accepts all strings that start in \(q_0\) and end in a final state.

**Example:** \(L = \{a^n b^n \mid n \geq 0\}\), \(\Sigma = \{a, b\}\), \(\Gamma = \{z, a\}\)

---

**Another Definition for Language Acceptance**

NPDA \(M\) accepts \(L(M)\) by empty stack:

\[L(M) = \{ w \in \Sigma^* \mid (q_0, w, z)^* \vdash (p, \lambda, \lambda) \}\]
Example: \( L = \{ a^n b^m c^{n+m} | n, m > 0 \}, \Sigma = \{ a, b, c \}, \Gamma = \{ 0, z \} \)

Example: \( L = \{ ww^R | w \in \Sigma^+ \}, \Sigma = \{ a, b \}, \Gamma = \{ z, a, b \} \)

Example: \( L = \{ ww | w \in \Sigma^* \}, \Sigma = \{ a, b \} \)

Examples for you to try on your own: (solutions are at the end of the handout).

- \( L = \{ a^n b^m | m > n, m, n > 0 \}, \Sigma = \{ a, b \}, \Gamma = \{ z, a \} \)
- \( L = \{ a^n b^{n+m} c^m | n, m > 0 \}, \Sigma = \{ a, b, c \} \),
- \( L = \{ a^n b^{2n} | n > 0 \}, \Sigma = \{ a, b \} \)
**Definition:** A PDA $M=(Q,\Sigma,\Gamma,\delta,q_0,z,F)$ is deterministic if for every $q \in Q$, $a \in \Sigma \cup \{\lambda\}$, $b \in \Gamma$

1. $\delta(q,a,b)$ contains at most 1 element
2. if $\delta(q,\lambda,b) \neq \emptyset$ then $\delta(q,c,b) = \emptyset$ for all $c \in \Sigma$

**Definition:** $L$ is DCFL iff $\exists$ DPDA $M$ s.t. $L=L(M)$.

**Examples:**

1. Previous pda for $\{a^n b^n | n \geq 0\}$ is deterministic?
2. Previous pda for $\{a^n b^m c^{n+m} | n, m > 0\}$ is deterministic?
3. Previous pda for $\{ww^R | w \in \Sigma^+\}, \Sigma = \{a, b\}$ is deterministic?
Example: $L=\{a^n b^m | m > n, n, m > 0\}$, $\Sigma = \{a, b\}$, $\Gamma = \{z, a\}$

Example: $L=\{a^n b^n c^m | n, m > 0\}$, $\Sigma = \{a, b, c\}$

Example: $L=\{a^n b^{2n} | n > 0\}$, $\Sigma = \{a, b\}$