Section: Other Models of Turing Machines

Definition: Two automata are equivalent if they accept the same language.

Turing Machines with Stay Option

Modify $\delta$,

Theorem Class of standard TM’s is equivalent to class of TM’s with stay option.

Proof:

$\Rightarrow$: Given a standard TM $M$, then there exists a TM $M'$ with stay option such that $L(M) = L(M')$. 
• ($\iff$): Given a TM $M$ with stay option, construct a standard TM $M'$ such that $L(M) = L(M')$.

$M = (Q, \Sigma, \Gamma, \delta, q_0, B, F)$

$M' =$

For each transition in $M$ with a move (L or R) put the transition in $M'$. So, for

$$\delta(q_i, a) = (q_j, b, L \text{ or } R)$$

put into $\delta'$

For each transition in $M$ with S (stay-option), move right and move left. So for

$$\delta(q_i, a) = (q_j, b, S)$$

$L(M) = L(M')$. QED.
Definition: A *multiple track* TM divides each cell of the tape into $k$ cells, for some constant $k$.

A 3-track TM:

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tape head

A multiple track TM starts with the input on the first track, all other tracks are blank.

$\delta$: 


Theorem Class of standard TM’s is equivalent to class of TM’s with multiple tracks.

Proof: (sketch)

• \((\Rightarrow)\): Given standard TM M there exists a TM M’ with multiple tracks such that \(L(M) = L(M')\).

• \((\Leftarrow)\): Given a TM M with multiple tracks there exists a standard TM M’ such that \(L(M) = L(M')\).
Definition: A TM with a semi-infinite tape is a standard TM with a left boundary.

Theorem Class of standard TM’s is equivalent to class of TM’s with semi-infinite tapes.

Proof: (sketch)

• ($\Rightarrow$): Given standard TM $M$ there exists a TM $M'$ with semi-infinite tape such that $L(M) = L(M')$. Given $M$, construct a 2-track semi-infinite TM $M'$
\( (\Leftarrow): \) Given a TM \( M \) with semi-infinite tape there exists a standard TM \( M' \) such that \( L(M) = L(M') \).
Definition: An Multitape Turing Machine is a standard TM with multiple (a finite number) read/write tapes.

For an n-tape TM, define \( \delta \):
Theorem Class of Multitape TM’s is equivalent to class of standard TM’s.

Proof: (sketch)

• (⇐): Given standard TM M, construct a multitape TM M’ such that L(M)=L(M’).

• (⇒): Given n-tape TM M construct a standard TM M’ such that L(M)=L(M’).
Definition: An Off-Line Turing Machine is a standard TM with 2 tapes: a read-only input tape and a read/write output tape.

Define $\delta$: 

![Diagram showing the control unit and tapes with symbols a, b, c, and b, b, d]
Theorem Class of standard TM’s is equivalent to class of Off-line TM’s.

Proof: (sketch)

• ($\Rightarrow$): Given standard TM $M$ there exists an off-line TM $M'$ such that $L(M)=L(M')$.

• ($\Leftarrow$): Given an off-line TM $M$ there exists a standard TM $M'$ such that $L(M)=L(M')$. 

```
#  a  b  c
# 1
# b b d
# 1
```

↑
Running Time of Turing Machines

Example:

Given \( L = \{ a^n b^n c^n | n > 0 \} \). Given \( w \in \Sigma^* \), is \( w \) in \( L \)?

Write a 3-tape TM for this problem.
Definition: An Multidimensional-tape Turing Machine is a standard TM with a multidimensional tape

\[
\begin{array}{ccc}
\uparrow \\
| & | & | \\
| & a & b \\
| & | & | \\
\downarrow
\end{array}
\]

Define \( \delta \):
Theorem Class of standard TM’s is equivalent to class of 2-dimensional-tape TM’s.

Proof: (sketch)

• \((\Rightarrow)\): Given standard TM M, construct a 2-dim-tape TM M’ such that \(L(M) = L(M’)\).

• \((\Leftarrow)\): Given 2-dim tape TM M, construct a standard TM M’ such that \(L(M) = L(M’)\).
Construct $M'$
Definition: A *nondeterministic* Turing machine is a standard TM in which the range of the transition function is a set of possible transitions.

Define $\delta$:

Theorem Class of deterministic TM’s is equivalent to class of nondeterministic TM’s.

Proof: (sketch)

$\bullet$ ($\Rightarrow$): Given deterministic TM $M$, construct a nondeterministic TM $M'$ such that $L(M) = L(M')$.

$\bullet$ ($\Leftarrow$): Given nondeterministic TM $M$, construct a deterministic TM $M'$ such that $L(M) = L(M')$. Construct $M'$ to be a 2-dim tape TM.
A step consists of making one move for each of the current machines. For example: Consider the following transition:

\[ \delta(q_0, a) = \{(q_1, b, R), (q_2, a, L), (q_1, c, R)\} \]

Being in state \( q_0 \) with input abc.
The one move has three choices, so 2 additional machines are started.

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Definition: A 2-stack NPDA is an NPDA with 2 stacks.

Define $\delta$:  

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stack 2

stack 1
Consider the following languages which could not be accepted by an NPDA.

1. \( L = \{ a^n b^n c^n | n > 0 \} \)

2. \( L = \{ a^n b^n a^n b^n | n > 0 \} \)

3. \( L = \{ w \in \Sigma^* | \) number of a’s equals number of b’s equals number of c’s},
\( \Sigma = \{ a, b, c \} \)
Theorem Class of 2-stack NPDA’s is equivalent to class of standard TM’s.

Proof: (sketch)

• (⇒): Given 2-stack NPDA, construct a 3-tape TM $M'$ such that $L(M) = L(M')$. 
\textbullet\ \leftarrow:\text{ Given standard TM }M, \text{ construct a 2-stack NPDA }M' \text{ such that } L(M) = L(M').
Universal TM - a programmable TM

● Input:
  – an encoded TM M
  – input string w

● Output:
  – Simulate M on w
An encoding of a TM

Let TM \( M = \{Q, \Sigma, \Gamma, \delta, q_1, B, F\} \)

- \( Q = \{q_1, q_2, \ldots, q_n\} \)
  
  Designate \( q_1 \) as the start state.
  
  Designate \( q_2 \) as the only final state.
  
  \( q_n \) will be encoded as \( n \) 1’s

- Moves
  
  L will be encoded by 1
  
  R will be encoded by 11

- \( \Gamma = \{a_1, a_2, \ldots, a_m\} \)
  
  where \( a_1 \) will always represent the B.
For example, consider the simple TM:

\[
\begin{align*}
\delta(q_1, a) &= (q_1, a, R), \\
\delta(q_1, b) &= (q_2, a, L)
\end{align*}
\]

which can be represented as 5-tuples:

\[
(q_1, a, q_1, a, R), (q_1, b, q_2, a, L)
\]

Thus, the encoding of the TM is:

0101101011011010111011011010
For example, the encoding of the TM above with input string “aba” would be encoded as:

0101101101101101101101101101001101110110

Question: Given $w \in \{0, 1\}^+$, is $w$ the encoding of a TM?
Universal TM

The Universal TM (denoted $M_U$) is a 3-tape TM:
Program for $M_U$

1. Start with all input (encoding of TM and string $w$) on tape 1. Verify that it contains the encoding of a TM.

2. Move input $w$ to tape 2

3. Initialize tape 3 to 1 (the initial state)

4. Repeat (simulate TM $M$)
   (a) consult tape 2 and 3, (suppose current symbol on tape 2 is $a$ and state on tape 3 is $p$)
   (b) lookup the move (transition) on tape 1, (suppose $\delta(p,a)=(q,b,R)$.)
   (c) apply the move
       • write on tape 2 (write $b$)
       • move on tape 2 (move right)
       • write new state on tape 3 (write $q$)
Observation: Every TM can be encoded as string of 0’s and 1’s.

Enumeration procedure - process to list all elements of a set in ordered fashion.

Definition: An infinite set is countable if its elements have 1-1 correspondence with the positive integers.

Examples:

- \( S = \{ \text{positive odd integers} \} \)
- \( S = \{ \text{real numbers} \} \)
- \( S = \{ w \in \Sigma^+ \}, \Sigma = \{ a, b \} \)
- \( S = \{ \text{TM’s} \} \)
- \( S = \{ (i,j) \mid i,j > 0, \text{are integers} \} \)
Linear Bounded Automata

We place restrictions on the amount of tape we can use,

\[
\begin{array}{c}
  [a b c] \\
  \uparrow
\end{array}
\]

Definition: A linear bounded automaton (LBA) is a nondeterministic TM \( M=(Q,\Sigma, \Gamma, \delta, q_0, B, F) \) such that \([,] \in \Sigma\) and the tape head cannot move out of the confines of \([\cdot]’s. Thus, \( \delta(q_i, [) = (q_j, [, R), \text{ and } \delta(q_i, ]) = (q_j, ], L) \)

Definition: Let \( M \) be a LBA. 
\( L(M) = \{ w \in (\Sigma - \{[,\})^*|q_0[w] \vdash [x_1q_fx_2] \} \)

Example: \( L = \{a^n b^n c^n | n > 0 \} \) is accepted by some LBA