Maps

COMPSCI 230 — Discrete Math

February 2, 2017
Outline

1. Maps and Functions
Maps and Functions

• Maps and functions describe associations
• Most things that do something can be viewed as maps or functions
• Programs are either maps or functions, and associate inputs to outputs
• A map may be non-deterministic: same input $\rightarrow$ different outputs
• The Google search engine is a nondeterministic map from queries to lists of results
• A function is a deterministic map: same input $\rightarrow$ same output
• Sphere volume is a function of radius
• ...
Maps and Functions

• Maps are so general that it is hard to say anything about them
• Functions are constrained maps, and even more constraints lead to mathematically interesting objects
• Cool application: the size of infinite sets (Georg Cantor, 1845-1918)
• The set of reals and the power set of the naturals have equal size

Get to use fancy typefaces: There are $\aleph_0$ naturals and $\mathfrak{c}$ reals
• “Aleph-zero” is arabic ’A’ and “cee” is German Fraktur ’c’
• Get to prove that $\aleph_0 - 1 = \aleph_0$. This is good for parties:

$\aleph_0$ bottles of beer on the wall,
$\aleph_0$ bottles of beer.
Take one down, and pass it around,
$\aleph_0$ bottles of beer on the wall
(repeat)

Clicker Check

I am taking 230 because ...

(Not Graded)

A: I have to
B: My friends do
C: I’m interested in the material
D: I love pain
E: Other
Which relation holds if $S = \{0, 1, 2, 3\}$ and $T = \{2, 1, 3, 0, 1\}$?

Pick one

A: $S \in T$
B: $S \subset T$
C: $S \supseteq T$
D: All of the above
E: None of the above
Power Set

What is the cardinality of the power set of \( A = \{0, 1\} \)?

Pick one

<p>| | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>A</td>
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<tr>
<td>B</td>
<td>2</td>
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<tr>
<td>C</td>
<td>3</td>
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<tr>
<td>D</td>
<td>4</td>
</tr>
<tr>
<td>E</td>
<td>5</td>
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</tbody>
</table>

(Graded)
If $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$, what is $A \setminus B$?

Pick one (Graded)

A: $\emptyset$
B: $\{1, 3\}$
C: $\{1, 6\}$
D: $\{2, 4\}$
E: $\{1, 2, 3, 4, 6\}$
### Maps

A map

- **Domain** $D$
- **Codomain** $C$
- Some $x$s map to some $y$s
- There may be both gaps and duplicates in either column
- $\exists x \in D, \exists y \in C : y = f(x)$
Maps and Functions

Not a function

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
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<tbody>
<tr>
<td>a</td>
<td>p</td>
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<tr>
<td>b</td>
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A map is exactly $D$, $C$, and a nonempty subset of $D \times C$

Two problems prevent this map from being a function:

- Some $x \in D = \{a, b, c, d\}$ have no $f(x)$
- Some $x \in D$ have more than one $f(x)$
- OK that some $y$ is not matched
- OK that different $x$s are matched to the same $y$
A function from $D$ into $C$

$D = \{a, b, c\}, \ C = \{p, q, r, s\}$

- A function assigns exactly one $y \in C$ to every $x \in D$
- All $x \in D$ show up in the first column exactly once
- $\forall x \in D, \exists y \in C : y = f(x)$ (at least one) and
  $\forall x, x' \in D, y, y' \in C : y = f(x), \ y' = f(x'), \ y \neq y' \Rightarrow x \neq x'$ (at most one)
- All functions are “into”
Functions

Let $D = \{1, 2, 3, 4\}$ and $C = \{2, 4, 6\}$ and $f = \{(2, 4), (3, 4), (4, 2)\}$

Is $(D, C, f)$ a function? (Not Graded)

A: No, it is neither a function nor a map
B: No, but it is a map
C: Yes
A function but not an injection

\[ D = \{a, b, c\}, \quad C = \{p, q, r, s\} \]

Some \( y \in C \) are associated to more than one \( x \in D \)
Injections

An injection

• $D = \{a, b, c\}, \ C = \{p, q, r, s\}$
• An injection is a function that assigns a different $y \in C$ to every $x \in D$
• Function, plus distinct $x$s in the first column match distinct $y$s in the second
• Function, plus $\forall x, x' \in D : x \neq x' \Rightarrow f(x) \neq f(x')$
• “One-to-one function” is synonym for “injection”
Injections

Let $D = \{1, 2, 3, 4\}$ and $C = \{2, 4, 6\}$ and 
$f = \{(2, 4), (3, 2), (4, 6)\}$

Is $(D, C, f)$ an injection? (Not Graded)

A: No, it is not even a function
B: No, but it is a function
C: Yes
Functions and Surjections

A function but not a surjection

\[ D = \{a, b, c\}, \quad C = \{p, q, r\} \]

- Some \( y \in C \) are unassigned
Surjections

A surjection

\[ D = \{ a, b, c \}, \quad C = \{ p, q \} \]

- A surjection is function that assigns every \( y \) to some \( x \)
- Function, plus every \( y \in C \) shows up in the second column
- Function, plus \( \forall y \in C, \exists x \in D : y = f(x) \)
- “Onto function” is synonym for “surjection”
Functions

Let $D = \{1, 2, 3, 4\}$ and $C = \{2, 4, 6\}$ and $f = \{(1, 2), (2, 4), (3, 2), (4, 6)\}$

$(D, C, f)$ is...

A: Not a map
B: A map but not a function
C: An injection
D: A surjection
E: A function, but neither B nor C

(Not Graded)
Injections and Bijections

An injection but not a bijection

$$D = \{a, b, c\}, \quad C = \{p, q, r, s\}$$

- Function and one-to-one, but some \(y \in C\) are unassigned
- Some \(y\)s in the second column have no matching \(x\)
Bijections

A bijection

$D = \{a, b, c\}, C = \{p, q, r\}$

- A bijection has exactly one $y \in C$ for every $x \in D$ and vice versa.
- All $x \in D$ show up in the first column exactly once, all $y \in C$ show up in the second column exactly once, no gaps.
- $\text{Bijection} = \text{injection} \cap \text{surjection}$
- “One-to-one and onto function” is synonym for “bijection”
Let $D = \{1, 2, 3, 4\}$ and $C = \{2, 4, 6\}$. How many bijections are there between $D$ and $C$?

Pick one...

A: 0  
B: 1  
C: $2^3$  
D: $4 \cdot 3$  
E: Infinitely many
Bijections are Invertible

\[ D = \{a, b, c\}, \quad C = \{p, q, r\} \]

- Bijection = injection \cap surjection
- Bijection = one-to-one \cap onto
- Bijection = injection both ways
- \(|D| = |C|\) (same cardinality)
- This is where Cantor comes in...
$D \subseteq \mathbb{R}, C \subseteq \mathbb{R}$, Continuous $f$

- **Map**
- **Function**
- **Injection**
- **Surjection**
- **Bijection**