

Homework 0: Probability background (due January 17 before class)

Please read the rules for assignments on the course web page (<https://www2.cs.duke.edu/courses/spring20/compsci323d/>). Use Piazza (preferred) or directly contact Hyoungh-Yoon (hk236@duke.edu), Jiali (jx76@cs.duke.edu), Caspar (cpo11@duke.edu) or Vince (conitzer@cs.duke.edu) with any questions. This short assignment is intended to give you (and us) an opportunity to assess your background in probability theory, which you will need later in the course.

1 Conditional probability

We have three different toys, a red one (r), a blue one (b) and a green one (g). Children have preferences over these toys, which determine which one they choose to play with when offered a choice between two of them. For example, the ordering $b \succ g \succ r$ indicates a preference for the blue over the green, the green over the red, and (by transitivity) the blue over the red. No child is ever indifferent between two toys. All preference orderings are equally likely. For example, there are just as many children with preferences $b \succ g \succ r$ as there are with preferences $r \succ g \succ b$.

Now imagine that we offer a child with unknown preferences two of the toys, say the red and the blue one. The child picks one, say, the blue one, and plays with it. After a while the child gets bored with the toy and so we offer the two remaining toys, i.e., the green and the red one. **What** is the probability that in this second round the child will choose to play with the toy that we already offered in the first round (i.e., the red one)? Try to use formal notation in your answer, e.g., $P(r \succ g \mid b \succ r)$.

2 Probability density functions

Now assume we have three bidders in a one-shot auction: Hyoungh-Yoon, Jiali, and Caspar. They bid independently and simultaneously on the item according to their valuations (i.e., they bid their true values, unless constrained by their budgets), which follow certain probability density functions (PDF).

2.1 Calculate expectation of bid

Assume Jiali's valuation for the item is drawn from a uniform PDF: $v_J \sim U[1, 3]$, but he only has a budget of 2. He cannot bid higher than his budget. Thus, his bid will be $b_J = \min(v_J, 2)$. **What** is the expected value of his bid b_J ?

2.2 Calculate probability

The rule of the auction is easy: whoever bids highest wins. Assume that Hyung-Yoon's bid b_H and Caspar's bid b_C have no budget constraints and are drawn independently from a uniform distribution: $b_H, b_C \sim U[0, 3]$. **What** is then the cumulative distribution function (CDF) of the highest bid between Hyung-Yoon and Caspar? (Recall that the CDF gives the probability that a random variable is below a certain value. Thus, this CDF gives the probability that both their bids are below that value.)

Now suppose that Jiali, for some reason, does not get to evaluate the item, so he simply bids his expected value $\mathbb{E}[v_J]$. **What** is the probability that Jiali wins the auction with $\mathbb{E}[v_J]$?

Bonus question: **What** is the probability that Jiali wins, if he does evaluate the item, and bids b_J (keep in mind the budget constraint) instead of $\mathbb{E}[v_J]$?

3 Combinatorics

We have 10 students with all different heights. Suppose they have to sit in a single line of 10 chairs, behind each other, facing a board. We are hoping that shorter students always sit in front of taller ones, so that everyone can see the board clearly. Suppose the seats are initially assigned randomly (uniformly at random). Then, we want the students to switch seats to sit in order. **What** is the probability that the following number of students have to change their seats? You may use expressions such as $10!$ and $\binom{10}{2}$ in your answer.

- No one.
- Exactly one student. (There is a simple commonsense answer to this.)
- Exactly two students.