

Recitation Week 6: Kidney exchange problem with additional modeling considerations

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- Q: Recall the standard integer programming formulation for the kidney exchange problem, first without constraints on the lengths of cycles.

A1: We begin with a compatibility graph, where an edge from node i to node j indicates that patient i wants donor j 's kidney. Therefore, note that each node in this graph represents a donor-patient pair.

In this graph, suppose C is the set of all (simple) cycles and x_c is a binary variable that encodes whether all edges in cycle c are used. Then the problem is formulated as follows:

$$\begin{aligned} \max \quad & \sum_{c \in C} |c| x_c \\ \text{subject to} \quad & \sum_{c: v \in c} x_c \leq 1 \text{ (for each vertex } v) \end{aligned}$$

A2: Another formulation: For each edge, make a binary variable x_{ij} .

$$\begin{aligned} & \max \sum_{ij} x_{ij} \\ & \text{subject to } \sum_j x_{ij} = \sum_j x_{ji} \text{ (for each } i) \\ & \sum_i x_{ij} \leq 1 \text{ (for each } j) \end{aligned}$$

- Q: How would you incorporate constraining the length of cycles to some integer k ?

A1: If you use the set of cycles as a parameter, then you can simply let C only be the set of cycles of length $\leq k$?

A2: In the other formulation, you would need the following additional constraint:

$$\sum_{i \leq j \leq k} x_{i_j i_{j+1}} \leq k - 1 \text{ (for every path } i_1, i_2, \dots, i_k, i_{k+1} \text{ with } i_1 \neq i_{k+1})$$

Alternatively, in lecture there was a solution using events/operations.

- Q: Some people might have multiple willing donors, but all of them might be incompatible. When entering the kidney exchange, we might imagine that we want to allow them to list multiple possible donors to increase the chance of being matched. We here imagine that (for various reasons) patients cannot give more than one donor in any particular solution. How can this be modeled?

A: Actually, nothing about the linear program itself needs to be changed. We can simply change the input. For example, in the compatibility graph, draw an edge from node i to node j if patient i is compatible with at least one of j 's donors. So having multiple donors is just like having one very widely compatible donor.

- Q: How can the models that we considered deal with altruistic donors (donors without an intended recipient)?

A: In the compatibility graph, add a node for each altruistic donor. Add edges from any altruistic donor to each node j whose patient could receive the altruistic donor's kidney. The rest depends on which ILP we used earlier.

In the cycle model: With altruistic donors, we are now not only looking for (simple) cycles, but also for (simple) paths. So we do the same as before but now C consists of two kinds of things: cycles involving only donor-patient pairs and paths that start with an altruistic donor and then only have donor-patient pairs. In each case $|c|$ must denote the number of recipients. The ILP works as before.

In the other model, you need to change the constraint $\sum_j x_{ij} = \sum_j x_{ji}$ to $\sum_j x_{ij} \geq \sum_j x_{ji}$ and make sure that it is only imposed on donor-patient pairs i and not on altruistic donors. The former change has to be made because there might now be chains in which the last donor patient pair only gets a kidney. Also, you need to add $\sum_i x_{ji} \leq 1$ for each j to make sure that no altruistic donor gives more than one kidney.

- Q: In the real world, only about 10 percent of matches lead to actual transplants due to logistical issues, newly discovered incompatibilities, etc. How can our problem formulation be improved to address such issues?

A: One can imagine various ways to improve the model. First, put a weight w_e on each edge to represent the value of the edge being chosen and a probability of success p_e on each edge. Then, we can create a new parameter $u_c = \sum_{e \in c} w_e \cdot \prod_{e \in c} p_e$. Finally, update the objective function to

$$\max \sum_{c \in C} u_c x_c.$$