COMPSCI 323 - Computational Microeconomics

Recitation Week 7: Lung exchanges and properties of voting rules

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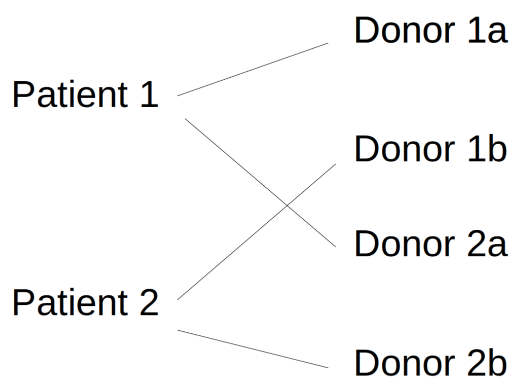
1 Lung exchanges

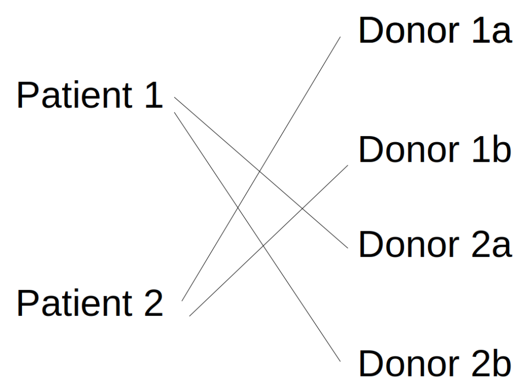
1.(a) The idea

See Ergin et al. (2015): Lung Exchange. Same idea as kidney exchanges. But lungs are different from kidneys in important ways. A healthy lung consists of five lobes. A patient with a dysfunctional lung will usually need two lobes. A donor will usually only give one lobe. Hence, a patient needs two donors. As with kidneys, there are compatibility requirements (related to blood type but in the case of lungs also size). Hence, there are patients with two willing donors who still cannot get their two lobes. The goal of a lung exchange is therefore to allow donors to swap donors, thereby increasing the number of patients who get their dysfunctional lung replaced.

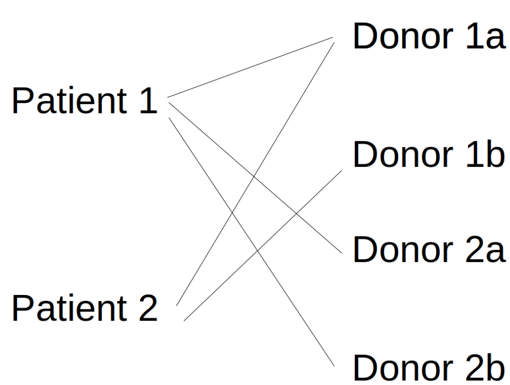
1.(b) An example

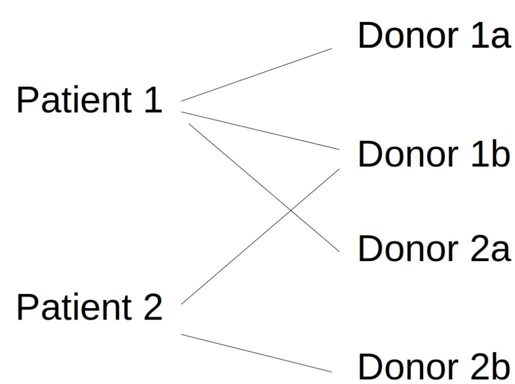
The simplest possible example is a swap between two patient-donor triplets, in which either one or both donors are swapped:



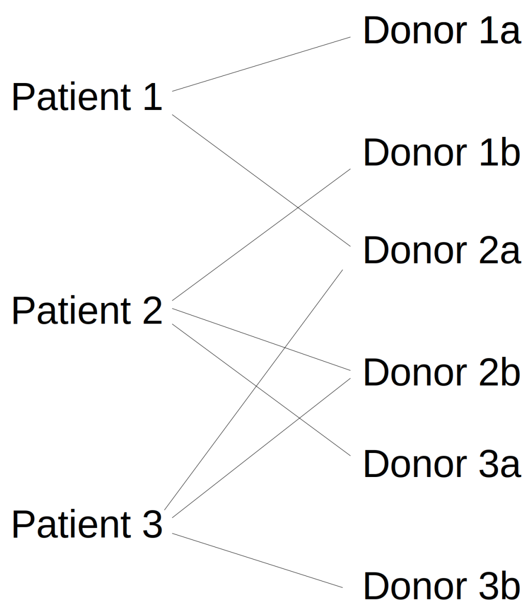


How about the following?





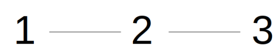
Here’s a more complicated example:



1.(c) Algorithmic questions

- Q: For the same reasons as with kidney exchanges, one might imagine that we want to limit the size of "cycles", or in this case sets of dependent transplants. Let's say we wanted to limit it to 2. That is, there can only be swaps between two patient-donor triplets. Then we might consider the problem of finding the optimal set of transplants to make. Can this be solved efficiently, i.e., in polynomial time?

  A: Yes! Just as we did in class (for kidney exchanges), we can reduce the problem to the problem of finding maximum matchings in general (potentially non-bipartite) graphs. That problem can be solved in polynomial time. For each patient-donor triplet, create a node. Connect two nodes, if the corresponding triplets can do any donor swap. For example, the above example with three patients gives the following graph:



- Q: What if we restrict the cycle size to some value that is greater than 2?

  A: It's NP-complete. As you learned in class (and as shown by Abraham et al. (2007): Clearing Algorithms for Barter Exchange Markets:Enabling Nationwide Kidney Exchanges, Theorem 1) this problem is hard even for kidney exchanges. Lung exchanges are at least as difficult as kidney exchanges (at least theoretically). For any kidney exchange problem, you can consider a corresponding lung exchange problem in which every patient brings one donor who is compatible only with that patient. And then another donor whose compatibility is like that of the kidney in the original kidney exchange problem.

- Another question is what happens if no restriction on the cycle size is imposed. Recall that for kidney exchanges, that problem can be solved in polynomial time! But it turns out that for lung exchanges, this problem is NP-complete. See Luo and Tang (2015): Mechanism Design and Implementation for Lung Exchange.

2 Properties of voting rules

Recently, Alice has become interested in the phenomenon of votes "cancelling out." Let us say that a set[1] S of votes *cancels out with respect to voting rule* r if for **every** set T of votes, the winner[2] that r produces for T is the same as the winner that r produces for $S \cup T$. For example, the set of votes $\{a \succ b \succ c, b \succ a \succ c, c \succ a \succ b\}$ cancels out with respect to the plurality rule: each candidate is ranked first once in this set of votes, so it has no net effect on the outcome of the election. The same set does not cancel out with respect to Borda, though, because from these votes, a gets 4 points, b gets 3, and c gets 2, which may affect the outcome of the election. Alice likes to know when a set of votes cancels out with respect to a rule, so that she can just ignore these votes, easing her computation of the winner.

2.(a)

Define a pair of *opposite votes* to be a pair of votes with completely opposite rankings of the candidates, i.e. the votes can be written as $c_1 \succ c_2 \succ \ldots \succ c_m$ and $c_m \succ c_{m-1} \succ \ldots \succ c_1$. Let us say that a voting rule r satisfies the *Opposites Cancel Out (OCO)* criterion if every pair of opposite votes cancels out with respect to r.

From among the (reasonable[3]) voting rules discussed in class, give 3 voting rules that satisfy the OCO criterion, and 3 that do not (and say which ones are which!).

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[1] Technically, a multiset, since the same vote may occur multiple times.

[2] … or set of winners if there are ties.

[3] E.g., not dictatorial rules, rules for which there is a candidate that can't possibly win, randomized rules, etc. Also, approval cannot be one of the rules because it is not based on rankings. If you use Cup, Cup only satisfies a criterion if it satisfies it for every way of pairing the candidates.

2.(b)

Define a *cycle* of votes to be a set of votes that can be written as $c_1 \succ c_2 \succ \ldots \succ c_m, c_2 \succ c_3 \succ \ldots c_m \succ c_1, c_3 \succ c_4 \succ \ldots \succ c_m \succ c_1 \succ c_2, \ldots, c_m \succ c_1 \succ c_2 \succ \ldots \succ c_{m-1}$. Let us say that a voting rule r satisfies the *Cycles Cancel Out (CCO)* criterion if every cycle cancels out with respect to r.

From among the (reasonable) voting rules discussed in class, give 3 voting rules that satisfy the CCO criterion, and 3 that do not.

2.(c)

Define a pair of *opposite cycles* of votes to be a cycle, plus all the opposite votes of votes in that cycle. Note that these opposite votes themselves constitute a cycle, the opposite of which is the original cycle. Let us say that a voting rule r satisfies the *Opposite Cycles Cancel Out (OCCO)* criterion if every pair of opposite cycles cancels out with respect to r.

From among the (reasonable) voting rules discussed in class, give 5 voting rules that satisfy the OCCO criterion, and 1 that does not.

2.(d)

Criterion C_1 is *stronger* than criterion C_2 if every rule that satisfies C_1 also satisfies C_2. Two criteria are *incomparable* if neither is stronger than the other. For every pair of criteria among OCO, CCO, and OCCO, say which one is stronger (or that they are incomparable).

3 Answers

3.(a) 3 voting rules that satisfy the OCO criterion, and 3 that do not

(i) OCO

Since adding a pair of opposite votes actually doesn't change the pairwise election result (and pairwise election graph, consequently), theoretically all voting rules based on pairwise election is OCO.

To see why, let's prove it briefly:

Consider a pair of votes with completely opposite rankings of the m candidates:

$c_1 \succ c_2 \succ \ldots \succ c_i \succ \ldots \succ c_j \succ \ldots \succ c_m$ and $c_m \succ c_{m-1} \succ \ldots \succ c_j \succ \ldots \succ c_i \succ \ldots \succ c$

Easy to see that for any pair of candidates c_i and c_j, such that $i \neq j$, they must tie in terms of pairwise election in these two votes. Hence this pair of votes has no net effect on all pairwise election results. For instance, Copeland, Maximin, and Slater are OCO.

(ii) Not OCO

Plurality, veto and STV are not OCO. To see why, we give this example:

- 2 votes $a \succ b \succ c$
- 2 votes $b \succ a \succ c$

Before adding votes, a and b tie under plurality and veto rules.

After we add a pair of opposite votes: $b \succ a \succ c$ and $c \succ a \succ b$, a wins with respect to veto, but b wins with respect to plurality.

Let's give a counter-example of STV that violate OCO:

Consider a pair of opposite votes: $a \succ b \succ c \succ d$ and $d \succ c \succ b \succ a$. And a bundle of votes with STV rules:

- 2 votes of $a \succ \dots$
- 2 votes of $d \succ \dots$
- 2 votes of $b \succ a \succ \dots$

After adding $a \succ b \succ c \succ d$ and $d \succ c \succ b \succ a$, we change the outcome of this STV election from "a, b and d tie" to "a wins".

3.(b) 3 voting rules that satisfy the CCO criterion, and 3 that do not

(i) CCO

Yet adding a cycle of votes increases the times of a candidate being ranked first (and last) by one for each candidate. Therefore it does cancel out with respect to plurality and veto.

Similarly, in a cycle of votes Borda gives each candidate $1+2+\ldots+(m-1)$ points. Since every candidate gets same point, it has no net effect on the outcome of the Borda election.

(ii) Not CCO

$$\begin{aligned} &c_1 \succ c_2 \succ \ldots \succ c_m, \\ &c_2 \succ c_3 \succ \ldots c_m \succ c_1, \\ &c_3 \succ c_4 \succ \ldots \succ c_m \succ c_1 \succ c_2, \\ &\ldots \\ &c_m \succ c_1 \succ c_2 \succ \ldots \succ c_{m-1} \end{aligned} \tag{1}$$

Since adding a cycle of votes (Equation 1) does change the pairwise election result dramatically[4], so all voting rules based on pairwise election is not CCO, such as Copeland, Slater, and Kemeny.[5]

[4] For example, see c_1 and c_m, in that cycle of votes, c_1 only wins c_m once, but loses $m-1$ times. Therefore the pairwise election outcome between c_1 and c_m will change $m-2$.

[5] Notice that STV doesn't satisfy CCO. If c_1 and c_2 tie in the beginning, after adding the aforesaid cycle (Equation 1) c_1 will win.

3.(c) 5 voting rules that satisfy the OCCO criterion, and 1 that does not

(i) OCCO

Because an opposite cycles of votes has both the properties of a cycles of votes and a pair of opposite of votes, the similar reasoning can be used again:

Consider it as two cycle of votes, then it increases the times of a candidate being ranked first (and last) by two for each candidate. Therefore it does cancel out with respect to plurality and veto.

Consider it as many pairs of opposite votes, then it doesn't change the pairwise election result (and pairwise election graph, consequently). Therefore it does cancel out with respect to all voting rules based on pairwise election, e.g. Copeland, Maximin, and Slater.

(ii) Not OCCO

I suggest that STV is not OCCO since STV is neither OCO nor CCO. Here's a counter example:

- 1 votes of $a \succ d \succ \dots$
- 1 votes of $a \succ b \succ \dots$
- 2 votes of $d \succ \dots$
- 2 votes of $b \succ \dots$

Where we can see that a, b and d tie with respect to STV. However after we add a pair of opposite cycles of votes extended from $a \succ b \succ c \succ d$, i.e.:

$$\begin{aligned}
&a \succ b \succ c \succ d,\\
&d \succ c \succ b \succ a,\\
&b \succ c \succ d \succ a,\\
&a \succ d \succ c \succ b,\\
&c \succ d \succ a \succ b,\\
&b \succ a \succ d \succ c,\\
&d \succ a \succ b \succ c,\\
&c \succ b \succ a \succ d
\end{aligned}$$

c will be eliminated in the first round, and c's vote go to b and d. So will a in the second round. Then b and d tie.

3.(d) For every pair of criteria among OCO, CCO, and OCCO, say which one is stronger

(i) OCO and CCO

From subsection 3.(a) and subsection 3.(b) we see that there are some voting rules satisfy OCO but not CCO (e.g., Copeland, Maximin, and Slater), and vice versa (e.g., plurality and veto). Therefore, OCO and CCO are incomparable.

(ii) OCO and OCCO

Then we briefly prove that OCO is stronger than OCCO, which means every rule that satisfies OCO also satisfies OCCO.

Every rule that satisfies OCO cancels out with every pair of opposite votes. Because a pair of opposite cycles of votes is nothing more than many pairs of opposite votes, it (a pair of opposite cycles of votes) must cancel out with respect to OCO rules.

(iii) CCO and OCCO

Similarly, CCO is stronger than OCCO, which means every rule that satisfies CCO also satisfies OCCO.

Every rule that satisfies CCO cancels out with every cycle of votes. Because a pair of opposite cycles of votes is nothing more than two cycles of votes, it must cancel out with respect to CCO rules.